Unit 1: Classification of signals and systems

Signal

Signal is one that carries information and is defined as a physical quantity that varies with one or more independent variable.

Example: Music, speech

Classification of signals

Analog and Digital signal

Analog signal:

A signal that is defined for every instants of time is known as analog signal. Analog signals are continuous in amplitude and continuous in time. It is denoted by x(t). It is also called as **Continuous time** signal. Example for Continuous time signal is shown in Fig 1.1

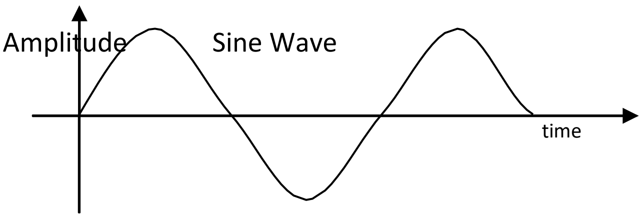


Fig 1.1 Continuous time signal

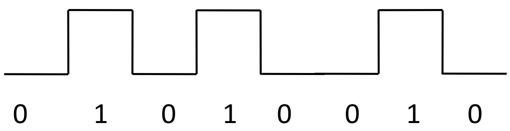


Fig 1.2 Digital Signal

Digital signal:

The signals that are discrete in time and quantized in amplitude is called digital signal (Fig 1.2)

Continuous time and discrete time signal

Continuous time signal:

A signal that is defined for every instants of time is known as continuous time signal. Continuous time signals are continuous in amplitude and continuous in time. It is denoted by x(t) and shown in Fig 1.1

Discrete time signal:

A signal that is defined for discrete instants of time is known as discrete time signal. Discrete time signals are continuous in amplitude and discrete in time. It is also obtained by sampling a continuous time signal. It is denoted by x(n) and shown in Fig 1.3

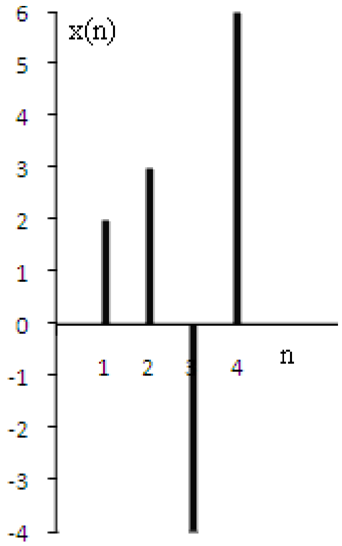


Fig 1.3 Discrete time signal

Even (symmetric) and Odd (Anti-symmetric) signal

Continuous domain:

Even signal:

A signal that exhibits symmetry with respect to t=0 is called even signal
Even signal satisfies the condition $x(t) = x(-t)$

Odd signal:

A signal that exhibits anti-symmetry with respect to t=0 is called odd signal
Odd signal satisfies the condition $x(t) = -x(-t)$

Even part $x_e(t)$ and Odd part $x_0(t)$ of continuous time signal $x(t)$:

Even part $x_e(t) = \frac{1}{2}[x(t) + x(-t)]$

Odd part $x_o(t) = \frac{1}{2}[x(t) - x(-t)]$

Discrete domain:

Even signal:

A signal that exhibits symmetry with respect to n=0 is called even signal
Even signal satisfies the condition $x(n) = x(-n)$.

Odd signal:

A signal that exhibits anti-symmetry with respect to n=0 is called odd signal Odd signal satisfies the condition $x(n) = -x(-n)$.

Even part $x_e(n)$ and Odd part $x_0(n)$ of discrete time signal $x(n)$:

Even part $x_e(n) = \frac{1}{2}[x(n) + x(-n)]$

Odd part $x_o(n) = \frac{1}{2}[x(n) - x(-n)]$

Periodic and Aperiodic signal

Periodic signal:

A signal is said to periodic if it repeats again and again over a certain period of time.

Aperiodic signal:

A signal that does not repeat at a definite interval of time is called aperiodic signal.

Continuous domain:

A Continuous time signal is said to **periodic** if it satisfies the condition

$$x(t) = x(t+T) \quad \text{where } T \text{ is fundamental time period}$$

If the above condition is not satisfied then the signal is said to be **aperiodic**

Fundamental time period $\mathbf{T} = \frac{2\pi}{\Omega}$, where Ω is fundamental angular frequency in rad/sec

Discrete domain:

A Discrete time signal is said to **periodic** if it satisfies the condition

$$x[n] = x[n+N] \quad \text{where } N \text{ is fundamental time period}$$

If the above condition is not satisfied then the signal is said to be **aperiodic**

Fundamental time period $\mathbf{N} = \frac{2\pi m}{\omega}$, where ω is fundamental angular frequency in rad/sec, m is smallest positive integer that makes N as positive integer

Energy and Power signal

Energy signal:

The signal which has finite energy and zero average power is called energy signal. The non periodic signals like exponential signals will have constant energy and so non periodic signals are energy signals.

i.e., For energy signal, $0 < E < \infty$ and $P = 0$

For Continuous time signals,

$$Energy\ E = \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 dt$$

For Discrete time signals,

$$Energy\ E = \lim_{N\to\infty} \sum_{n=-N}^{N} |x(n)|^2$$

Power signal:

The signal which has finite average power and infinite energy is called power signal. The periodic signals like sinusoidal complex exponential signals will have constant power and so periodic signals are power signals.

i.e., For power signal, $0 < P < \infty$ and $E = \infty$

For Continuous time signals,

$$Average\ power\ P = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2\, dt$$

For Discrete time signals,

$$Average\ power\ P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

Deterministic and Random signals

Deterministic signal:

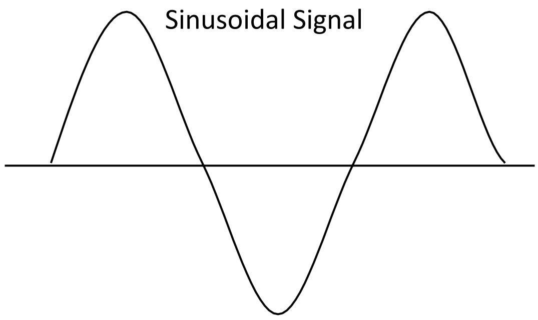
A signal is said to be deterministic if there is no uncertainity over the signal at any instant of time i.e., its instantaneous value can be predicted. It can be represented by mathematical equation.

Example: sinusoidal signal

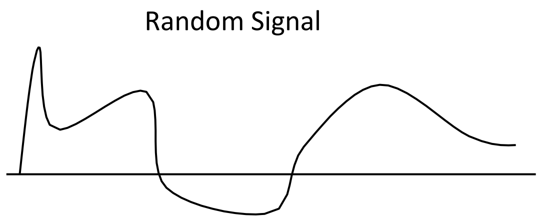
Random signal (Non-Deterministic signal):

A signal is said to be random if there is uncertainity over the signal at any instant of time i.e., its instantaneous value cannot be predicted. It cannot be represented by mathematical equation.

Example: noise signal



Deterministic signal



Random signal

Causal and Non-causal signal

Continuous domain:

Causal signal:

A signal is said to be causal if it is defined for t≥0.

$$i.e., \quad x(t) = 0 \quad for\ t < 0$$

Non-causal signal:

A signal is said to be non-causal, if it is defined for t< 0 or for both $t < 0$ and $t \geq 0$

$$i.e., \quad x(t) \neq 0 \quad for\ t < 0$$

When a non-causal signal is defined only for t<0, it is called as **anti-causal signal**

Discrete domain:

Causal signal:

A signal is said to be causal, if it is defined for n≥0.

$$i.e., \quad x(n) = 0 \quad for\, n < 0$$

Non-causal signal:

A signal is said to be non-causal, if it is defined for n< 0 or for both $n < 0$ *and* $n \geq 0$

$$i.e., \quad x(n) \neq 0 \quad for\, n < 0$$

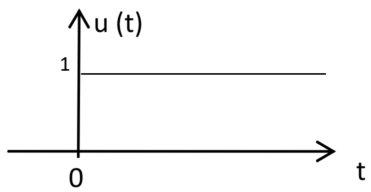
When a non-causal signal is defined only for n<0, it is called as **anti-causal signal**

Basic(Elementary or Standard) continuous time signals

Step signal

Unit Step signal is defined as

$$u(t) = 1 \; for\; t \geq 0$$
$$= 0 \; for\; t < 0$$

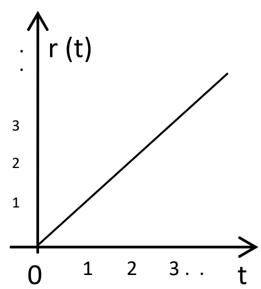


Unit step signal

Ramp signal

Unit ramp signal is defined as

$$r(t) = t \; for\; t \geq 0$$
$$= 0 \; for\; t < 0$$

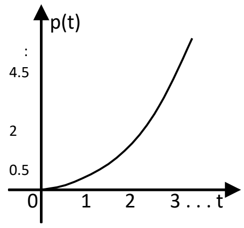


Unit ramp signal

Parabolic signal

Unit Parabolic signal is defined as

$$x(t) = \frac{t^2}{2} \; for \; t \geq 0$$
$$= 0 \; for \; t < 0$$



Unit Parabolic signal

Relation between Unit Step signal, Unit ramp signal and Unit Parabolic signal:

- Unit ramp signal is obtained by integrating unit step signal

$$i.e., \int u(t)\,dt = \int 1dt = t = r(t)$$

- Unti Parabolic signal is obtained by integrating unit ramp signal

$$i.e., \int r(t)\,dt = \int t\,dt = \frac{t^2}{2} = p(t)$$

- Unit step signal is obtained by differentiating unit ramp signal

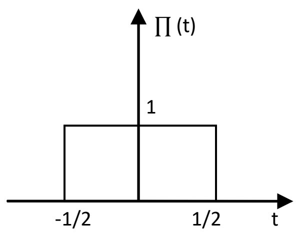
$$i.e., \frac{d}{dt} r(t) = \frac{d}{dt} t = 1 = u(t)$$

- Unit ramp signal is obtained by differentiating unit Parabolic signal

$$i.e., \frac{d}{dt} p(t) = \frac{d}{dt}\frac{t^2}{2} = \frac{1}{2}(2t) = t = r(t)$$

Unit Pulse signal is defined as

$$\prod(t) = 1 \; for \; |t| \leq \frac{1}{2}$$
$$= 0 \; elsewhere$$

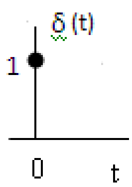


Unit Pulse signal

Impulse signal

Unit Impulse signal is defined as

$$\delta(t) = 0 \; for \; t \neq 0$$
$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1$$



Unit Impulse signal

Properties of Impulse signal: Property 1:

$$\int_{-\infty}^{\infty} \boldsymbol{x(t)\delta(t)\,dt = x(0)}$$

Proof:

$$\int_{-\infty}^{\infty} x(t)\delta(t)\,dt = x(0)\,\delta(0) = x(0) \qquad [\because \delta(t)\ exists\ only\ at\ t = 0\ and\ \delta(0) = 1]$$

Thus proved

Property 2:

$$\int_{-\infty}^{\infty} \boldsymbol{x(t)\delta(t - t_0)\,dt = x(t_0)}$$

Proof:

$$\int_{-\infty}^{\infty} x(t)\delta(t - t_0)\,dt = x(t_0)\,\delta(t_0 - t_0) = x(t_0)\,\delta(0) = x(t_0)$$

$$\because \delta(t - t_0)\ exists\ only\ at\ t = t_0\ and\ \delta(0) = 1$$

Thus proved

Sinusoidal signal

Cosinusoidal signal is defined as

$$x(t) = A\cos(\Omega t + \Phi)$$

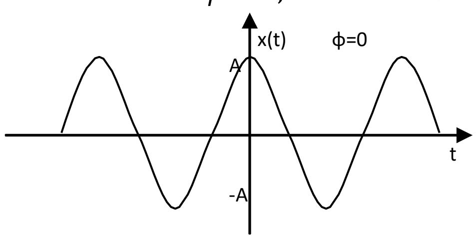
Sinusoidal signal is defined as

$$x(t) = A\sin(\Omega t + \Phi)$$

where $\Omega = 2\pi f = \frac{2\pi}{T}$ and Ω is angular frequency in rad/sec
f is frequency in cycles/sec or Hertz and
A is amplitude
T is time period in seconds
Φ is phase angle in radians

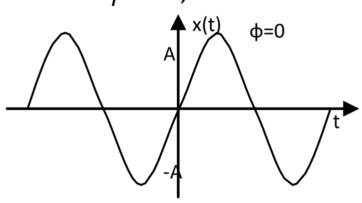
Cosinusoidal signal

$$when\ \phi = 0, x(t) = A\cos(\Omega t)$$



Sinusoidal signal

$$when\ \phi = 0, x(t) = A\sin(\Omega t)$$



Cosinusoidal signal

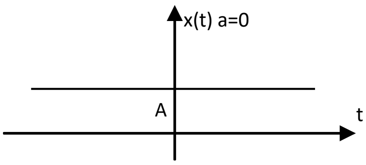
Sinusoidal signal

Exponential signal

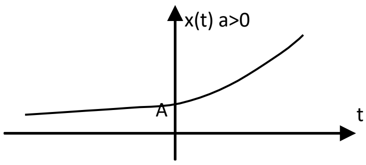
Real Exponential signal is defined as $x(t) = Ae^{at}$

where A is amplitude

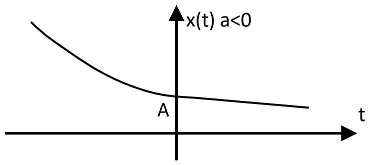
Depending on the value of 'a' we get dc signal or growing exponential signal or decaying exponential signal



DC signal



Exponentially growing signal



Exponentially decaying signal

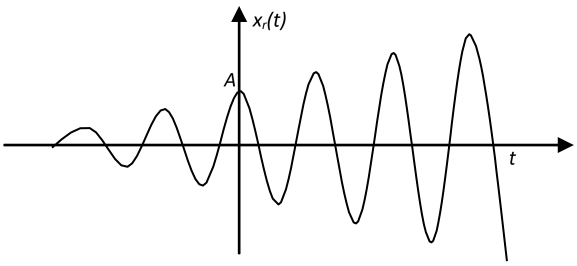
Complex exponential signal is defined as $x(t) = Ae^{st}$

where A is amplitude, s is complex variable and $s = \sigma + j\Omega$

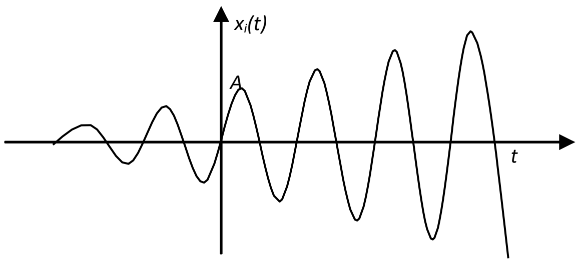
$$x(t) = Ae^{st} = Ae^{(\sigma + j\Omega)t} = Ae^{\sigma t}e^{j\Omega t} = Ae^{\sigma t}(cos\Omega t + jsin\Omega t)$$

$when\ \sigma = +ve, then\ x(t) = Ae^{\sigma t}(cos\Omega t + jsin\Omega t),$

$where\ x_r(t) = Ae^{\sigma t}cos\Omega t\ and\ x_i(t) = Ae^{\sigma t}sin\Omega t$



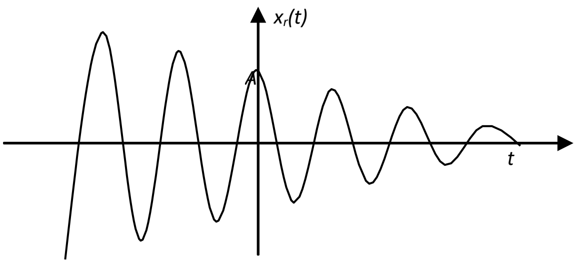
Exponentially growing Cosinusoidal signal

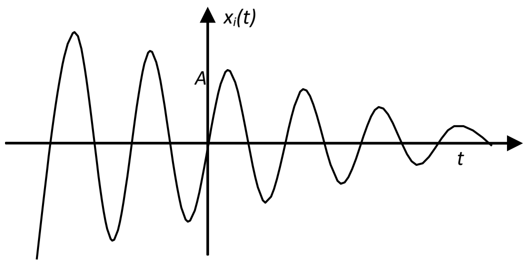


Exponentially growing sinusoidal signal

$when\ \sigma = -ve, then\ x(t) = Ae^{-\sigma t}(cos\Omega t + jsin\Omega t),$

$where\ x_r(t) = Ae^{-\sigma t}cos\Omega t\ and\ x_i(t) = Ae^{-\sigma t}sin\Omega t$





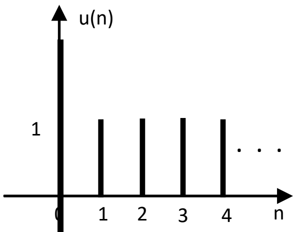
Exponentially decaying Cosinusoidal signal	Exponentially decaying sinusoidal signal

Basic(Elementary or Standard) Discrete time signals

Step signal

Unit Step signal is defined as

$$u(n) = 1 \ for\ n \geq 0$$
$$= 0 \ for\ n < 0$$

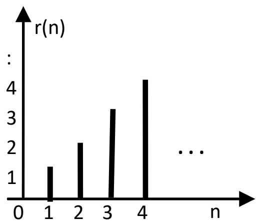


Unit step signal

Unit Ramp signal

Unit Ramp signal is defined as

$$r(n) = n \ for\ n \geq 0$$
$$= 0 \ for\ n < 0$$

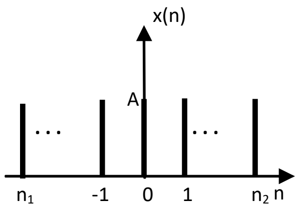


Unit Ramp signal

Pulse signal (Rectangular pulse function)

Pulse signal is defined as

$$x(n) = A \ for\ n_1 \leq n \leq n_2$$
$$= 0 \ elsewhere$$



Pulse signal

Unit Impulse signal

Unit Impulse signal is defined as

$$\delta(n) = 1 \; for \; n = 0$$
$$\delta(n) = 0 \; for \; n \neq 0$$

Unit Impulse signal

Sinusoidal signal

Cosinusoidal signal is defined as

$$x(n) = A\cos(\omega n)$$

Sinusoidal signal is defined as

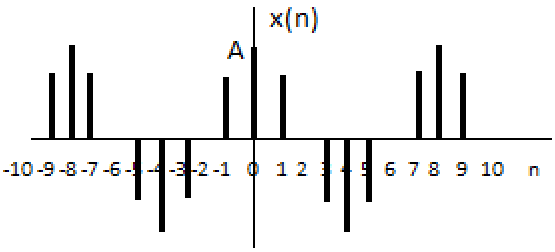
$$x(n) = A\sin(\omega n)$$

where $\omega = 2\pi f = \frac{2\pi}{N} m$ and ω is frequency in radians/sample

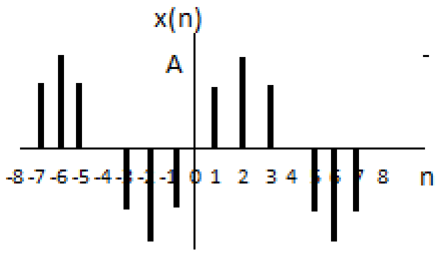
m is smallest integer

f is frequency in cycles/sample, A is amplitude

Cosinusoidal signal | **Sinusoidal signal**



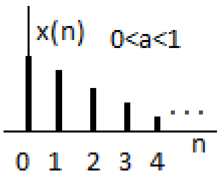
Cosinusoidal signal



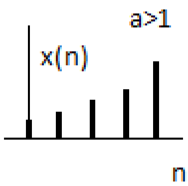
Sinusoidal signal

Exponential signal

Real Exponential signal is defined as $x(n) = a^n \; for \; n \geq 0$



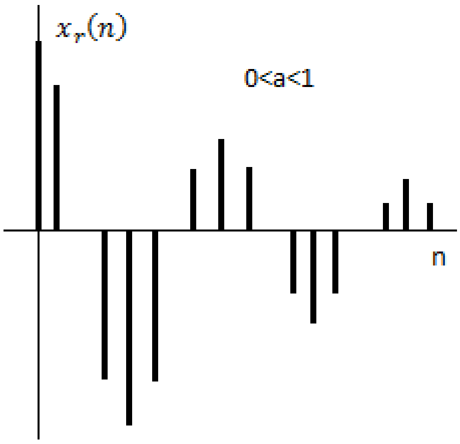
Decreasing exponential signal



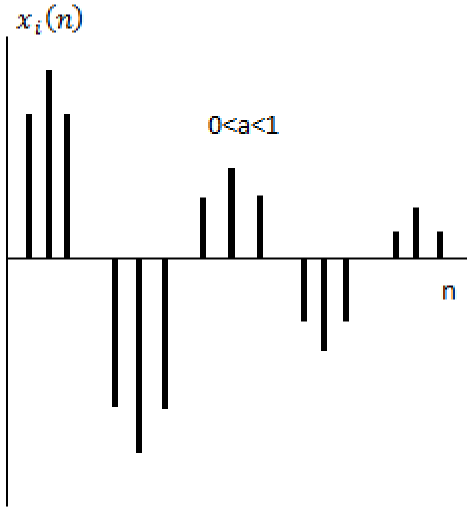
Increasing exponential signal

Complex Exponential signal is defined as $x(n) = a^n e^{j(\omega_0 n)} = a^n[\cos\omega_0 n + j\sin\omega_0 n]$

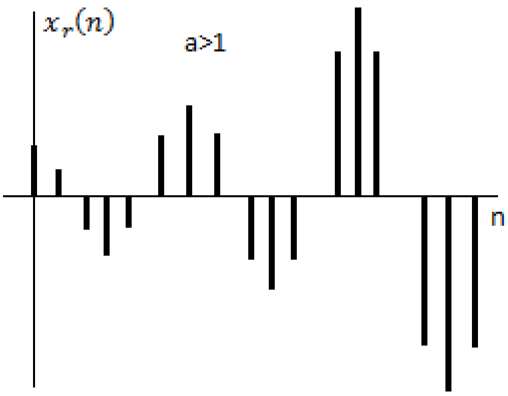
$$where \; x_r(n) = a^n\cos\omega_0 n \; and \; x_i(n) = a^n\sin\omega_0 n$$



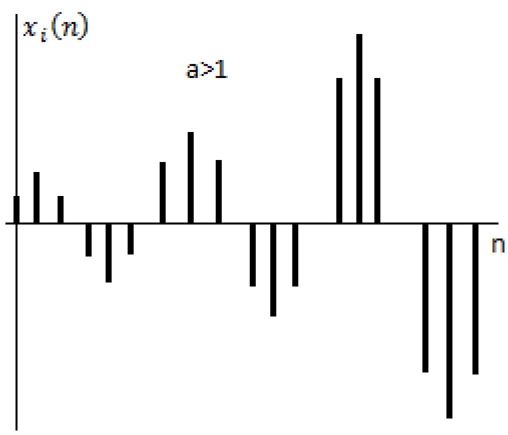
Exponentially decreasing Cosinusoidal signal



Exponentially decreasing sinusoidal signal



Exponentially growing Cosinusoidal signal



Exponentially growing sinusoidal signal

Classification of System

- Continuous time and Discrete time system
- Linear and Non-Linear system
- Static and Dynamic system
- Time invariant and Time variant system
- Causal and Non-Causal system
- Stable and Unstable system

Continuous time and Discrete time system

Continuous time system:

Continuous time system operates on a continuous time signal (input or excitation) and produces another continuous time signal (output or response) which is shown in Fig 1.84. The signal $x(t)$ is transformed by the system into signal $y(t)$, this transformation can be expressed as, Response $y(t) = T[x(t)]$

where $x(t)$ is input signal, $y(t)$ is output signal, and T denotes transformation

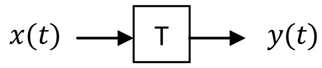


Fig 1.84 Representation of continuous time system

Discrete time system:

Discrete time system operates on a discrete time signal (input or excitation) and produces another discrete time signal (output or response) which is shown in Fig 1.85.

The signal $x(n)$ is transformed by the system into signal $y(n)$, this transformation can be expressed as,

Response $y(n) = T[x(n)]$

where x(n) is input signal, y(n) is output signal, and T denotes transformation

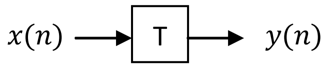


Fig 1.85 Representation of discrete time system

Linear system and Non Linear system

Continuous time domain:

Linear system:

A system is said to be linear if it obeys superposition theorem. Superposition theorem states that the response of a system to a weighted sum of the signals is equal to the corresponding weighted sum of responses to each of the individual input signals.

Condition for Linearity:

$$T[ax_1(t) + bx_2(t)] = ay_1(t) + by_2(t)$$

where $y_1(t)$ and $y_2(t)$ are the responses of $x_1(t)$ and $x_2(t)$ respectively

Non Linear system:

A system is said to be Non linear if it does not obeys superposition theorem.

$$i.e., T[ax_1(t) + bx_2(t)] \neq ay_1(t) + by_2(t)$$

where $y_1(t)$ and $y_2(t)$ are the responses of $x_1(t)$ and $x_2(t)$ respectively

Discrete time domain:

Linear system:

A system is said to be linear if it obeys superposition theorem. Superposition theorem states that the response of a system to a weighted sum of the signals is equal to the corresponding weighted sum of responses to each of the individual input signals.

Condition for Linearity:

$$T[ax_1(n) + bx_2(n)] = ay_1(n) + by_2(n)$$

where $y_1(n)$ and $y_2(n)$ are the responses of $x_1(n)$ and $x_2(n)$ respectively

Non Linear system:

A system is said to be Non linear if it does not obeys superposition theorem.

$$i.e., T[ax_1(n) + bx_2(n)] \neq ay_1(n) + by_2(n)$$

where y_1 n and y_2 n are the responses of x_1 n and x_2 n respectively

Static (Memoryless) and Dynamic (Memory) system

Continuous time domain:

Static system:

A system is said to be memoryless or static if the response of the system is due to present input alone.

Example: $y(t) = 2x(t)$

$y(t) = x^2(t) + x(t)$

Dynamic system:

A system is said to be memory or dynamic if the response of the system depends on factors other than present input also.

Example: $y(t) = 2x(t) + x(-t)$

$y(t) = x^2(t) + x(2t)$

Discrete time domain:

Static system:

A system is said to be memoryless or static if the response of the system is due to present input alone.

Example: $y(n) = x(n)$

$y(n) = x^2(n) + 3x(n)$

Dynamic system:

A system is said to be memory or dynamic if the response of the system depends on factors other than present input also.

Example: $y(n) = 2x(n) + x(-n)$

$y(n) = x^2(1-n) + x(2n)$

Time invariant (Shift invariant) and Time variant (Shift variant) system

Continuous time domain:

Time invariant system:

A system is said to time invariant if the relationship between the input and output does not change with time.

If $y\ t = T\ x\ t$

Then $T\ x\ t - t_0 = y(t - t_0)$ should be satisfied for the system to be time invariant

Time variant system:

A system is said to time variant if the relationship between the input and output changes with time.

If $y\ t = T\ x\ t$

Then $T\ x\ t - t_0 \neq y(t - t_0)$ should be satisfied for the system to be time variant

Discrete time domain:

Time invariant system:

A system is said to time invariant if the relationship between the input and output does not change with time.

If $y\ n = T\ x\ n$

Then $T\ x\ n - n_0 = y(n - n_0)$ should be satisfied for the system to be time invariant

Time variant system:

A system is said to time variant if the relationship between the input and output changes with time.

If $y\ n = T\ x\ n$

Then $T\ x\ n - n_0 \neq y(n - n_0)$ should be satisfied for the system to be time variant

Causal and Non-Causal system

Continuous time domain:

Causal system:

A system is said to be causal if the response of a system at any instant of time depends only on the present input, past input and past output but does not depends upon the future input and future output.

Example: $y(t) = 3x(t) + x(t-1)$

A system is said to be causal if impulse response $h(t)$ is zero for negative values of t i.e., $h(t) = 0\ for\ t < 0$

Non-Causal system:

A system is said to be Non-causal if the response of a system at any instant of time depends on the future input and also on the present input, past input, past output.

Example: $y(t) = x(t+2) + x(t-1)$

$y(t) = x(-t) + x(t+4)$

A system is said to be non-causal if impulse response $h(t)$ is non-zero for negative values of t i.e., $h(t) \neq 0\ for\ t < 0$

Discrete time domain:

Causal system:

A system is said to be causal if the response of a system at any instant of time depends only on the present input, past input and past output but does not depends upon the future input.

Example: $y(n) = 3x(n) + x(n-1)$

A system is said to be causal if impulse response h(n) is zero for negative values of n i.e., $h(n) = 0\ for\ n < 0$

Non-Causal system:

A system is said to be Non-causal if the response of a system at any instant of time depends on the future input and also on the present input, past input, past output.

Example: $y(n) = x(n+2) + x(n-1)$

$$y(n) = x(-n) + x(n+4)$$

A system is said to be non-causal if impulse response $h(n)$ is non-zero for negative values of n i.e., $h(n) \neq 0$ for $n < 0$

Stable and Unstable system

Continuous time domain:

A system is said to be **stable** if and only if it satisfies the BIBO stability criterion.

BIBO stable condition:

- Every bounded input yields bounded output.

$i.e.,$ if $0 < x(t) < \infty$ then $0 < y(t) < \infty$ should be satisfied for the system to be stable

- Impulse response should be absolutely integrable

$$i.e., 0 < \int_{-\infty}^{\infty} h(\tau)\, d\tau < \infty$$

If the BIBO stable condition is not satisfied, then the system is said to be **unstable** system

Discrete time domain:

A system is said to be **stable** if and only if it satisfies the BIBO stability criterion.

BIBO stable condition:

- Every bounded input yields bounded output.
- Impulse response should be absolutely summable

$$i.e., 0 < \sum_{k=-\infty}^{\infty} h(k) < \infty$$

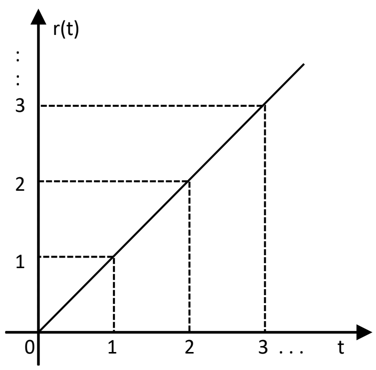
If the BIBO stable condition is not satisfied, then the system is said to be **unstable** system

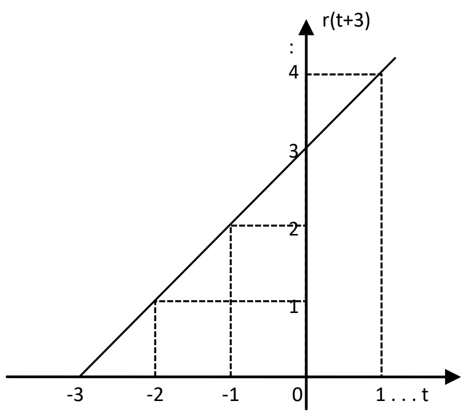
Solved Problems

1. Draw $r(t+3)$, where r(t) is ramp signal

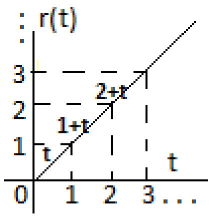
Solution:

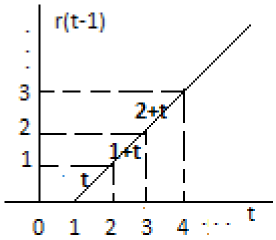
$r(t) = t; t \geq 0$

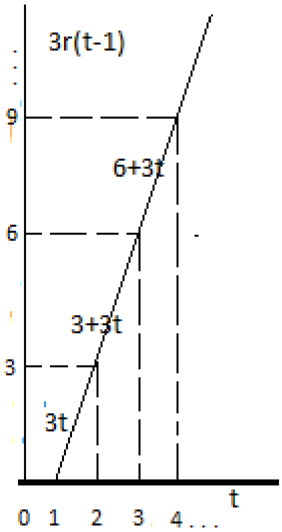


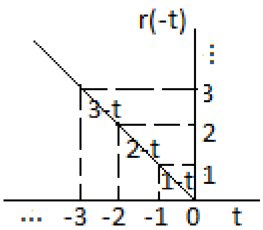


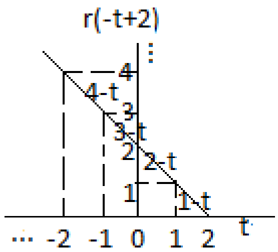
2. Sketch $x(t) = 3r(t-1) + r(-t+2)$











$x\,t = 3r\,t - 1 + r(-t+2)$

$= 0 + 4 - t \; for -2 \le t \le -1$

$= 0 + 3 - t \; for -1 \le t \le 0$

$= 0 + 2 - t \; for \; 0 \le t \le 1$

$= 3t + 1 - t \; for \; 1 \le t \le 2$

$= 3 + 3t + 0 \; for \; 2 \le t \le 3$

$= 6 + 3t + 0 \; for \; 3 \le t \le 4$

and so on

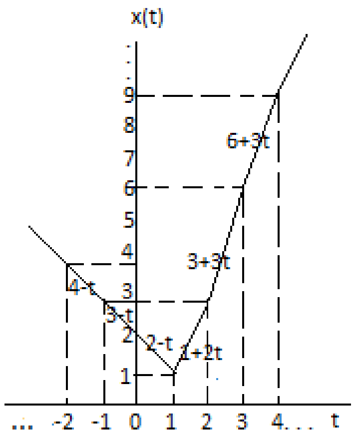
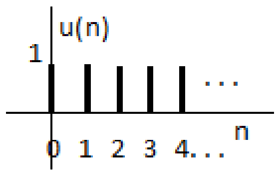


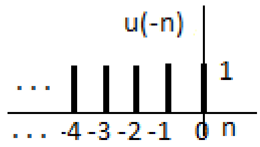
Fig 1.163

3. Draw time reversal signal of unit step signal

Solution:

$u\,n = 1; n \ge 0$





4. Check whether the following is periodic or not. If periodic, determine fundamental time period

a. $\boldsymbol{x\ t = 2\cos 5t + 1 - \sin 4t}$

Here $\Omega_1 = 5, \Omega_2 = 4$

$$T_1 = \frac{2\pi}{\Omega_1} = \frac{2\pi}{5} = \frac{2\pi}{5}$$

$$T_2 = \frac{2\pi}{\Omega_2} = \frac{2\pi}{4} = \frac{\pi}{2}$$

$$\frac{T_1}{T_2} = \frac{\frac{2\pi}{5}}{\frac{\pi}{2}} = \frac{4}{5} \text{ (It is rational number)}$$

Hence $x(t)$ is **periodic**

$$T = 5T_1 = 4T_2 = 2\pi$$

$\therefore x(t)$ is **periodic** with period $\boldsymbol{2\pi}$

b. $x\ n = 3\cos 4\pi n + 2\sin \pi n$

Here $\omega_1 = 4\pi, \omega_2 = \pi$

$$N_1 = \frac{2\pi m}{\omega_1} = \frac{2\pi m}{4\pi} = \frac{m}{2}$$

$N_1 = 1$ (taking $m = 2$)

$$N_2 = \frac{2\pi m}{\omega_2} = \frac{2\pi m}{\pi} = 2m$$

$N_2 = 2$ (taking $m = 1$)

$$N = LCM\ 1,2 = 2$$

Hence x(n) $\therefore x(n)$ is **periodic** with period **2**

5. Determine whether the signals are energy or power signal

$$\boldsymbol{x\ t = e^{-3t}u(t)}$$

$$\boldsymbol{Energy\ E_\infty} = \lim_{T\to\infty} \int_{-T}^{T} |x\ t|^2 dt = \lim_{T\to\infty} \int_{0}^{T} |e^{-3t}|^2 dt = \lim_{T\to\infty} \int_{0}^{T} e^{-6t}\, dt = \lim_{T\to\infty} \left[\frac{e^{-6t}}{-6}\right]_0^T$$

$$= \lim_{T\to\infty} \frac{e^{-6T}}{-6} - \frac{e^{-0}}{-6} = \frac{\mathbf{1}}{\mathbf{6}} < \infty \qquad \because e^{-\infty} = 0, e^{-0} = 1$$

$$\textbf{Power } P_\infty = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |x(t)|^2 dt = \lim_{T\to\infty} \frac{1}{2T}\int_{0}^{T} (e^{-3t})^2 dt = \lim_{T\to\infty} \frac{1}{2T}\int_{0}^{T} e^{-6t} dt$$

$$= \lim_{T\to\infty} \frac{1}{2T}\left[\frac{e^{-6t}}{-6}\right]_0^T = \lim_{T\to\infty} \frac{1}{2T}\left[\frac{e^{-6T}}{-6} - \frac{e^{-0}}{-6}\right] = \lim_{T\to\infty} \frac{1}{2T}\,\frac{1}{6} = \mathbf{0} \qquad \because e^{-\infty} = 0, e^{-0} = 1, \frac{1}{\infty} = 0$$

Since **energy** value is **finite** and average **power** is **zero,** the given signal is an **energy** signal.

6. Determine whether the signals are energy or power signal

$$\boldsymbol{x(n) = e^{j\left(\frac{\pi n}{4}+\frac{\pi}{2}\right)}}$$

$$\textbf{Energy } E_\infty = \lim_{N\to\infty}\sum_{n=-N}^{N} |x(n)|^2 = \lim_{N\to\infty}\sum_{n=-N}^{N} \left|e^{j\left(\frac{\pi n}{4}+\frac{\pi}{2}\right)}\right|^2 = \lim_{N\to\infty}\sum_{n=-N}^{N} 1^2 = \lim_{N\to\infty} 2N+1 = \infty$$

$$\because \left|e^{j(\omega n+\theta)}\right| = 1 \text{ and } \sum_{n=-N}^{N} 1 = 2N+1$$

$$\textbf{Average power } P_\infty = \lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{N} |x(n)|^2 = \lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{N} \left|e^{j\left(\frac{\pi n}{4}+\frac{\pi}{2}\right)}\right|^2$$

$$= \lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{N} 1^2 = \lim_{N\to\infty}\frac{1}{2N+1}\,(2N+1) = \mathbf{1}$$

Since **energy** value is **infinite** and average **power** is **finite,** the given signal is **power** signal

7. Determine whether the following systems are linear or not

$$\boldsymbol{\frac{dy(t)}{dt} + t\,y(t) = x^2(t)}$$

Output due to weighted sum of inputs:

$$\frac{d[ay_1(t) + by_2(t)]}{dt} + t[ay_1(t) + by_2(t)] = [ax_1(t) + bx_2(t)]^2 \ldots (1)$$

Weighted sum of outputs:

For input $x_1(t)$:

$$\frac{dy_1(t)}{dt} + t y_1(t) = x_1^2(t) \ldots (2)$$

For input $x_2(t)$:

$$\frac{dy_2(t)}{dt} + t y_2(t) = x_2^2(t) \ldots (3)$$

$$2\times a + 3\times b \Rightarrow a\frac{dy_1(t)}{dt} + aty_1(t) + b\frac{dy_2(t)}{dt} + bty_2(t) = ax_1^2(t) + bx_2^2(t) \ldots (4)$$

$$1 \neq (4)$$

The given system is **Non-Linear**

8. Determine whether the following systems are linear or not

$\boldsymbol{y\,n = x\,n - 2 + x(n^2)}$

Output due to weighted sum of inputs:

$$y_3\,n = ax_1\,n - 2 + bx_2\,n - 2 + ax_1\,n^2 + bx_2(n^2)$$

Weighted sum of outputs:

For input $x_1\,n$:

$$y_1\ n\ = x_1\ n - 2\ + x_1\ n^2$$

For input $x_2\,n$:

$$y_2\,n\ = x_2\,n - 2\ +\ x_2(n^2\,)$$

$$ay_1\,n + by_2\,n = ax_1\,n - 2 + ax_1\,n^2 + bx_2\,n - 2 + bx_2(n^2\,)$$

$$\because y_3\ n\ = ay_1\ n\ + by_2\,n$$

9. Determine whether the following systems are static or dynamic

$\boldsymbol{y\,t = x\,2t + 2x\,t}$

$$y\,0 = x\,0 + 2x\,0 \Rightarrow \text{present inputs}$$

$$y\,{-1} = x\,{-2} + 2x\,{-1} \Rightarrow \text{past and present inputs}$$

$$y\,1 = x\,2 + 2x\,1 \Rightarrow \text{future and present inputs}$$

Since output depends on past and future inputs the given system is **dynamic system**

10. Determine whether the following systems are static or dynamic

$\boldsymbol{y(n) = sinx(n)}$

$$y\ 0\ =\ sinx(0)\quad \Rightarrow\ \text{present input}$$

$$y\ {-1}\ =\ sinx(-1)\quad \Rightarrow\ \text{present input}$$

$$y\ 1\ =\ sinx(1)\quad \Rightarrow\ \text{present input}$$

Since output depends on present input the given system is **Staticsystem**

11. Determine whether the following systems are time invariant or not

$\boldsymbol{y(t) = x(t)sinwt}$

Output due to input delayed by T seconds

$$y(t,T) = x(t-T)sinwt$$

Output delayed by T seconds

$$y(t-T) = x(t-T)sinw(t-T)$$

$$\because y\,t,T \neq y\,t-T$$

The given system is **time variant**

12. Determine whether the following systems are time invariant or not

$\boldsymbol{y\,n = x(-n+2)}$

Output due to input delayed by k seconds

$$y\,n,k = x(-n+2-k)$$

Output delayed by k seconds

$$y\,n-k = x-(n-k+2) = x(-n+k+2)$$

$$\because\ \boldsymbol{y\ n,k\ \neq\ y\ n-k}$$

The given system is **time variant**

13. Determine whether the following systems are causal or not

$$y(t) = \frac{dx(t)}{dt} + 2x(t)$$

The given equation is differential equation and the output depends on past input. Hence the given system is **Causal**

14. Determine whether the following systems are causal or not

$$y(n) = \sin x(n)$$

$$y(0) = \sin x(0) \quad \Rightarrow \text{ present input}$$
$$y(-1) = \sin x(-1) \quad \Rightarrow \text{ present input}$$
$$y(1) = \sin x(1) \quad \Rightarrow \text{ present input}$$

Since output depends on present input the given system is **Causal system**

15. Determine whether the following systems are stable or not

$$h(t) = e^{-4t}u(t)$$

Condition for stability $\int_{-\infty}^{\infty} |h(\tau)|\, d\tau < \infty$

$$\int_{-\infty}^{\infty} |h(\tau)|\, d\tau = \int_{-\infty}^{\infty} e^{-4\tau}\, u(\tau)\, d\tau = \int_{0}^{\infty} e^{-4\tau}\, d\tau = \left[\frac{e^{-4\tau}}{-4}\right]_0^{\infty} = \frac{1}{4}$$

$\because \int_{-\infty}^{\infty} |h(\tau)|\, d\tau < \infty$ the given system is **stable**

16. Determine whether the following systems are stable or not

$$y(n) = 3x(n)$$

Let $x(n) = \delta(n)$, $y(n) = h(n)$

$$\Rightarrow h(n) = 3\delta(n)$$

Condition for stability $\sum_{k=-\infty}^{\infty} |h(k)| < \infty$

$$\sum_{k=-\infty}^{\infty} |h(k)| = \sum_{k=0}^{\infty} 3\delta(k) = \sum_{k=0}^{\infty} 3\delta(k) = 3$$

$\because \delta(k) = 0 \; for \; k \neq 0 \; and \; \delta(k) = 1 \; for \; k = 0$

$\because \sum_{k=-\infty}^{\infty} |h(k)| < \infty$ the given system is **stable**

Unit 2: Analysis of continuous time signals

Fourier series analysis

The Fourier representation of signals can be used to perform frequency domain analysis of signals in which we can study the various frequency components present in the signal, magnitude and phase of various frequency components.

Conditions for existence of Fourier series:
The Fourier series exist only if the following Dirichlet's conditions are satisfied.

- The signal $x(t)$ must be single valued function.
- The signal $x(t)$ must possess only a finite number of discontinuous in the period T.
- The signal must have a finite number of maxima and minima in the period T.
- $x(t)$ must be absolutely integrable. i.e., $\int_0^T |x(t)| dt < \infty$

Types of Fourier series:

- Trigonometric Fourier series
- Exponential Fourier series
- Cosine Fourier series

Trigonometric Fourier series

The trigonometric form of Fourier series of a periodic signal, $x(t)$ with period T is defined as

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\Omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\Omega_0 t \qquad \dots\dots\dots (1)$$

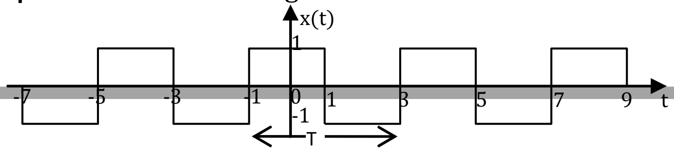
a_0, a_n, b_n → Fourier coefficients of trigonometric form of Fourier series

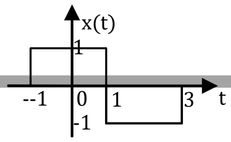
$$a_0 = \frac{1}{T} \int_{t_0}^{t_0+T} x(t) dt$$

$$a_n = \frac{2}{T} \int_{t_0}^{t_0+T} x(t) \cos n\Omega_0 t \, dt$$

$$b_n = \frac{2}{T} \int_{t_0}^{t_0+T} x(t) \sin n\Omega_0 t \, dt$$

Example 1 Find the trigonometric Fourier series for the periodic signal x(t) as shown in Figure





Solution:

$T = 3 - (-1) = 4 \; and \; \Omega_0 = \frac{2\pi}{T} = \frac{\pi}{2}$

Evaluation of a_0

$$\boldsymbol{a_0} = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)\,dt = \frac{1}{4}\left[\int_{-1}^{1} 1\,dt + \int_{1}^{3} -1\,dt\right] = \frac{1}{4}\left[[t]_{-1}^{1} - [t]_{1}^{3}\right] = \frac{1}{4}\left[(1-(-1)) - (3-1)\right]$$

$$= \frac{1}{4}[2-2] = \mathbf{0}$$

Evaluation of a_n

$$\boldsymbol{a_n} = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)\cos n\Omega_0 t\,dt = \frac{2}{4}\left[\int_{-1}^{1}\cos n\Omega_0 t\,dt + \int_{1}^{3}(-1)\cos n\Omega_0 t\,dt\right]$$

$$= \frac{1}{2}\left[\left[\frac{\sin n\Omega_0 t}{n\Omega_0}\right]_{-1}^{1} - \left[\frac{\sin n\Omega_0 t}{n\Omega_0}\right]_{1}^{3}\right] = \frac{1}{2}\left[\left[\frac{\sin n\frac{\pi}{2}t}{n\frac{\pi}{2}}\right]_{-1}^{1} - \left[\frac{\sin n\frac{\pi}{2}t}{n\frac{\pi}{2}}\right]_{1}^{3}\right]$$

$$= \frac{1}{2}\left(\frac{2}{n\pi}\right)\left[\sin n\frac{\pi}{2} - \left(\sin n\frac{\pi}{2}(-1)\right) - \left(\sin n\frac{\pi}{2}(3) - \sin n\frac{\pi}{2}\right)\right]$$

$$= \left[\frac{1}{n\pi}\right]\left[\sin n\frac{\pi}{2} + \sin n\frac{\pi}{2} - \sin 3n\frac{\pi}{2} + \sin n\frac{\pi}{2}\right] = \frac{1}{n\pi}\left[3\sin\frac{n\pi}{2} - \sin\left(2n\pi - \frac{n\pi}{2}\right)\right]$$

$$= \frac{1}{n\pi}\left[3\sin\frac{n\pi}{2} - \left(-\sin n\frac{\pi}{2}\right)\right] = \frac{\mathbf{4}}{\boldsymbol{n\pi}}\left[\boldsymbol{\sin n\frac{\pi}{2}}\right]$$

Evaluation of b_n

$$\boldsymbol{b_n} = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)\sin n\Omega_0 t\,dt = \frac{2}{4}\left[\int_{-1}^{1}\sin n\Omega_0 t\,dt + \int_{1}^{3} -\sin n\Omega_0 t\,dt\right]$$

$$= \frac{1}{2}\left[\left[\frac{-\cos n\Omega_0 t}{n\Omega_0}\right]_{-1}^{1} - \left[\frac{-\cos n\Omega_0 t}{n\Omega_0}\right]_{1}^{3}\right] = \frac{1}{2}\left[\left[\frac{-\cos n\frac{\pi}{2}t}{n\frac{\pi}{2}}\right]_{-1}^{1} + \left[\frac{\cos n\frac{\pi}{2}t}{n\frac{\pi}{2}}\right]_{1}^{3}\right]$$

$$= \frac{1}{2}\left[\frac{-2}{n\pi}\left(\cos n\frac{\pi}{2} - \cos n\frac{\pi}{2}(-1)\right) + \frac{2}{n\pi}\left(\cos n\frac{\pi}{2}(3) - \cos n\frac{\pi}{2}\right)\right]$$

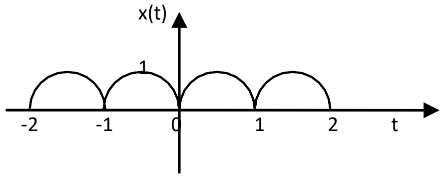
$$= \frac{1}{2}\left[0 + \frac{2}{n\pi}\left(\cos\left(2n\pi - \frac{n\pi}{2}\right) - \cos n\frac{\pi}{2}\right)\right] = \left[\frac{1}{n\pi}\left(\cos n\frac{\pi}{2} - \cos n\frac{\pi}{2}\right)\right] = \mathbf{0}$$

Trigonometric Fourier series

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n\cos n\Omega_0 t + \sum_{n=1}^{\infty} b_n\sin n\Omega_0 t$$

$$= \sum_{n=1}^{\infty}\frac{4}{n\pi}\sin\left(\frac{n\pi}{2}\right)\cos n\Omega_0 t = \sum_{n=1}^{\infty}\frac{4}{n\pi}\sin\left(\frac{n\pi}{2}\right)\cos n\frac{\pi}{2}t$$

Example 2 Obtain Fourier series of the following full wave rectified sine wave shown in figure



Solution:

$$x(t) = x(-t); \quad \therefore \textit{Given signal is even signal, so } \boldsymbol{b_n = 0}$$

$$T = 1 \textit{ and } \Omega_0 = \frac{2\pi}{1} = 2\pi$$

The given signal is sinusoidal signal, $\therefore x(t) = A \sin \Omega t$

$$\text{Here } \Omega = \frac{2\pi}{T} = \frac{2\pi}{2} = \pi \quad \text{and A} = 1$$

$$\therefore \boldsymbol{x(t) = \sin \pi t}$$

Evaluation of a_0

$$\boldsymbol{a_0} = \frac{2}{T}\int_0^{\frac{T}{2}} x(t)\, dt = \frac{2}{1}\int_0^{\frac{1}{2}} x(t)\, dt = 2\int_0^{\frac{1}{2}} \sin \pi t \, dt = 2\left[-\frac{\cos \pi t}{\pi}\right]_0^{\frac{1}{2}} = -\frac{2}{\pi}\left[\cos\frac{\pi}{2} - \cos 0\right] = \boldsymbol{\frac{2}{\pi}}$$

Evaluation of a_n

$$\boldsymbol{a_n} = \frac{4}{T}\int_0^{\frac{T}{2}} x(t) \cos n\Omega_0 t \, dt = \frac{4}{1}\int_0^{\frac{1}{2}} \sin \pi t \cos n2\pi t \, dt = 2\int_0^{\frac{1}{2}} [\sin((1+2n)\pi t) + \sin((1-2n)\pi t)]\, dt$$

$$= 2\left[-\frac{\cos((1+2n)\pi t)}{(1+2n)\pi} - \frac{\cos((1-2n)\pi t)}{(1-2n)\pi}\right]_0^{\frac{1}{2}}$$

$$= \frac{2}{\pi}\left[-\frac{\cos\left((1+2n)\frac{\pi}{2}\right)}{1+2n} - \frac{\cos\left((1-2n)\frac{\pi}{2}\right)}{1-2n} + \frac{1}{1+2n} + \frac{1}{1-2n}\right]$$

$$= \frac{2}{\pi}\left[\frac{1}{1+2n} + \frac{1}{1-2n}\right] = \frac{2}{\pi}\left[\frac{1-2n+1+2n}{1-4n^2}\right] = \boldsymbol{\frac{4}{\pi(1-4n^2)}}$$

Trigonometric Fourier series

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\Omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\Omega_0 t$$

$$\therefore x(t) = \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{4}{\pi(1-4n^2)} \cos n2\pi t$$

Exponential Fourier series

The exponential form of Fourier series of a periodic signal $x(t)$ with period T is defined as,

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\Omega_0 t}$$

The Fourier coefficient c_n can be evaluated using the following formulae

$$c_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)e^{-jn\,\Omega_0 t}\,dt$$

Example 3 Find exponential series for the signal shown in figure

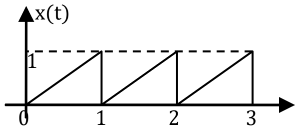


Fig 2.26

Solution:

$T = 1, \Omega_0 = \frac{2\pi}{T} = \frac{2\pi}{1} = 2\pi$

Consider the equation of a straight line

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1} \quad \ldots\ldots\ldots (9)$$

Consider one period of the given signal Fig 2.26 as shown in Fig 2.27Consider points P,Q as shown in fig 2.27

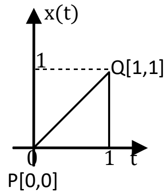


Fig 2.27

Coordinates of point $P = [0,0]$

Coordinates of point $Q = [1,1]$

On substituting the coordinates of points P and Q in eq (9)

$$\frac{x(t) - 0}{1 - 0} = \frac{t - 0}{1 - 0} \Rightarrow x(t) = t$$

$$[\because x = t, y = x(t)]$$

Evaluation of c_0

$$c_0 = \frac{1}{T}\int_0^T x(t)dt = \frac{1}{1}\int_0^1 (t)dt = \left[\frac{t^2}{2}\right]_0^1 = \frac{1}{2}$$

Evaluation of c_n

$$c_n = \frac{1}{T}\int_0^T x(t)e^{-jn\,\Omega_0 t}\,dt = \frac{1}{1}\int_0^1 te^{-jn\,2\pi t}\,dt = \left[t\frac{e^{-jn\,2\pi t}}{-jn2\pi}\right]_0^1 - \int_0^1 \frac{e^{-jn\,2\pi t}}{-jn2\pi}dt$$

$$= \frac{e^{-jn\,2\pi}}{-jn2\pi} + 0 + \left[\frac{e^{-jn\,2\pi t}}{-j^2(n2\pi)^2}\right]_0^1 = j\frac{e^{-jn\,2\pi}}{n2\pi} + \frac{e^{-jn\,2\pi}}{n^2 4\pi^2} - \frac{1}{n^2 4\pi^2}$$

$$= \frac{j}{n2\pi} + \frac{1}{n^2 4\pi^2} - \frac{1}{n^2 4\pi^2} = \frac{j}{n2\pi}$$

$$c_n = \frac{j}{n2\pi}$$

$c_1 = \frac{j}{2\pi}$, $c_2 = \frac{j}{4\pi}$, $c_3 = \frac{j}{6\pi}$, $c_{-1} = \frac{j}{-2\pi}$, $c_{-2} = \frac{j}{-4\pi}$, $c_{-3} = \frac{j}{-6\pi}$

Exponential Fourier series

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\,\Omega_0 t}$$

$$\therefore\ x(t) = +\cdots - \frac{j}{6\pi}e^{-j6\pi t} - \frac{j}{4\pi}e^{-j4\pi t} - \frac{j}{2\pi}e^{-j2\pi t} + \frac{1}{2} + \frac{j}{2\pi}e^{j2\pi t} + \frac{j}{4\pi}e^{j4\pi t} + \frac{j}{6\pi}e^{j6\pi t} + \cdots$$

$$= \frac{1}{2} + \frac{j}{2\pi}[e^{j2\pi t} - e^{-j2\pi t}] + \frac{j}{4\pi}[e^{j4\pi t} - e^{-j4\pi t}] + \frac{j}{6\pi}[e^{j6\pi t} - e^{-j6\pi t}] + \cdots$$

$$= \frac{1}{2} + \frac{1}{\pi}\left[\frac{e^{j2\pi t} - e^{-j2\pi t}}{(-1)2j}\right] + \frac{1}{2\pi}\left[\frac{e^{j4\pi t} - e^{-j4\pi t}}{(-1)2j}\right] + \frac{1}{3\pi}\left[\frac{e^{j6\pi t} - e^{-j6\pi t}}{(-1)2j}\right]$$

$$= \frac{1}{2} + \left(\frac{-1}{\pi}\right) \sin 2\pi t - \frac{1}{2\pi}\sin 4\pi t - \frac{1}{3\pi}\sin 6\pi t$$

$$= \frac{1}{2} - \frac{1}{\pi}\left[\sin 2\pi t + \frac{1}{2}\sin 4\pi t + \frac{1}{3}\sin 6\pi t + \cdots\right]$$

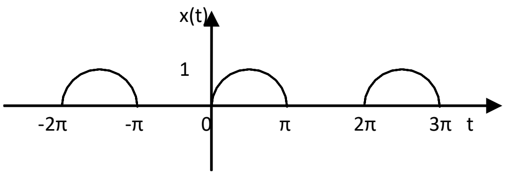
Cosine Fourier series

Cosine representation of $x(t)$ is

$$x(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\Omega_0 t + \theta_n)$$

Where A_0 is dc component, A_n is harmonic amplitude or spectral amplitude and θ_n is phase coefficient or phase angle *or spectral angle*

Example 4 Determine the cosine Fourier series of the signal shown in Figure



Solution:
The signal shown in is periodic with period $T = 2\pi$ *and* $\Omega_0 = \frac{2\pi}{2\pi} = 1$

The given signal is sinusoidal signal, $\therefore x(t) = A\sin\Omega t$

$$\text{Here}\ \ \Omega = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1,\ \ A = 1$$

$$\therefore \boldsymbol{x(t) = \sin t}$$

Evaluation of a_0

$$a_0 = \frac{1}{T}\int_0^T x(t)dt = \frac{1}{2\pi}\int_0^{\pi} \sin t\, dt = \frac{1}{2\pi}[-\cos t]_0^{\pi} = \frac{1}{2\pi}[-\cos\pi + \cos 0] = \frac{1}{2\pi}[2] = \frac{1}{\pi}$$

Evaluation of a_n

$$a_n = \frac{2}{T}\int_0^T x(t)\cos n\Omega_0 t\,dt = \frac{2}{2\pi}\int_0^\pi \sin t\,\cos nt\,dt = \frac{1}{2\pi}\int_0^\pi [\sin(1+n)t + \sin(1-n)t]\,dt$$

$$= \frac{1}{2\pi}\left[-\frac{\cos(1+n)t}{(1+n)} - \frac{\cos(1-n)t}{(1-n)}\right]_0^\pi$$

$$= \frac{1}{2\pi}\left[-\frac{\cos(1+n)\pi}{(1+n)} - \frac{\cos(1-n)\pi}{(1-n)} + \frac{1}{1+n} + \frac{1}{1-n}\right]$$

$$for\; n = odd : a_n = \frac{1}{2\pi}\left[-\frac{1}{1+n} - \frac{1}{1-n} + \frac{1}{1+n} + \frac{1}{1-n}\right] = 0$$

$$for\; n = even : a_n = \frac{1}{2\pi}\left[\frac{1}{1+n} + \frac{1}{1-n} + \frac{1}{1+n} + \frac{1}{1-n}\right] = \frac{1}{2\pi}\left(\frac{2}{1+n} + \frac{2}{1-n}\right)$$

$$= \frac{1}{\pi}\left[\frac{1-n+1+n}{1-n^2}\right] = \frac{2}{\pi(1-n^2)}$$

$$\therefore \boldsymbol{a_n} = \begin{cases} \mathbf{0} & \boldsymbol{for\; n = odd} \\ \dfrac{\mathbf{2}}{\boldsymbol{\pi}(\mathbf{1}-\boldsymbol{n}^{\mathbf{2}})} & \boldsymbol{for\; n = even} \end{cases}$$

Evaluation of b_n

$$\boldsymbol{b_n} = \frac{2}{T}\int_0^T x(t)\sin n\Omega_0 t\;dt = \frac{2}{2\pi}\int_0^\pi \sin t\;\sin nt\;dt = \frac{1}{2\pi}\int_0^\pi (\cos(1-n)t - \cos(1+n)t)\;dt$$

$$= \frac{1}{2\pi}\left[\frac{\sin(1-n)t}{(1-n)} - \frac{\sin(1+n)t}{(1+n)}\right]_0^\pi = \frac{1}{2\pi}\left[\frac{\sin(1-n)\pi}{(1-n)} - \frac{\sin(1+n)\pi}{(1+n)} - 0\right] = \mathbf{0}$$

Evaluation of Fourier coefficients of Cosine Fourier series from Trigonometric Fourier series:

$$A_0 = a_0 = \frac{1}{\pi}$$

$$A_n = \sqrt{a_n^{\,2} + b_n^{\,2}} = \frac{2}{\pi(1-n^2)},\; for\; n = even$$

$$\theta_n = -tan^{-1}\frac{b_n}{a_n} = 0$$

Cosine Fourier series

$$\mathrm{x(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\Omega_0 t + \theta_n)}$$

$$\therefore x(t) = \frac{1}{\pi} + \sum_{\substack{n=1\\(n=even)}}^{\infty} \frac{2}{\pi(1-n^2)}\cos nt = \frac{1}{\pi} + \frac{2}{\pi(1-4)}\cos 2t + \frac{2}{\pi(1-16)}\cos 4t + \cdots$$

$$= \frac{1}{\pi} - \frac{2}{3\pi}\cos 2t - \frac{2}{15\pi}\cos 4t + \cdots = \frac{1}{\pi} - \frac{2}{\pi}\left[\frac{1}{3}\cos 2t + \frac{1}{15}\cos 4t + \cdots\right]$$

Fourier transform

The Fourier representation of periodic signals has been extended to non-periodic signals by letting the fundamental period T tend to infinity and this Fourier method of representing non-periodic signals as a function of frequency is called Fourier transform.

Definition of Continuous time Fourier Transform

The Fourier transform (FT) of Continuous time signals is called Continuous Time Fourier Transform

$$Let\ x(t) = Continuous\ time\ signal$$
$$X(j\Omega) = F\{x(t)\}$$

The Fourier transform of continuous time signal, x(t) is defined as,

$$X(j\Omega) = F\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}\,dt$$

Conditions for existence of Fourier transform

The Fourier transform $x(t)$ exist if it satisfies the following Dirichlet condition

1. $x(t)$ should be absolutely integrable

$$ie\ ,\qquad \int_{-\infty}^{\infty} x(t)dt < \infty$$

2. $x(t)$ should have a finite number of maxima and minima with in any finite interval.
3. $x(t)$ should have a finite number of discontinuities with in any interval.

Definition of Inverse Fourier Transform

The inverse Fourier Transform of $X(j\Omega)$ is defined as,

$$x(t) = F^{-1}\{X(j\Omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\Omega)e^{j\Omega t}\,d\Omega$$

Example 5 Find Fourier transform of impulse signal

Solution:

By definition of Fourier transform

$$F\{x(t)\} = X(j\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}\,dt$$

$$\therefore F[\delta(t)] = \int_{-\infty}^{\infty} \delta(t)e^{-j\ \Omega t}\,dt$$

$$F[\delta(t)] = \delta(0)e^{-j\ \Omega(0)} = 1 \qquad [\because Impulse\ signal\ \delta(t) = \{\begin{matrix} 1\ for\ t = 0 \\ 0\ for\ t \neq 0 \end{matrix}]$$

Example 6 Find Fourier transform of double sided exponential signal

Solution:

Double sided exponential signal is given by

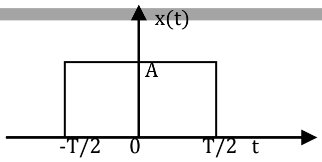
$$F[e^{-a|t|}] = \{\begin{matrix} e^{-at} & : t \geq 0 \\ e^{at} & : t \leq 0 \end{matrix}$$

$$F[e^{-a|t|}] = \int_{-\infty}^{0} e^{at}\,e^{-j\ \Omega t}\,dt + \int_{0}^{\infty} e^{-at}.e^{-j\ \Omega t}\,dt = \int_{-\infty}^{0} e^{(a-j\ \Omega)t}dt + \int_{0}^{\infty} e^{-(a+j\ \Omega)t}\,dt$$

$$= \left[\frac{e^{(a-j\ \Omega)t}}{a-j\,\Omega}\right]_{-\infty}^{0} + \left[\frac{e^{-(a+j\ \Omega)t}}{-(a+j\,\Omega)}\right]_{0}^{\infty} = \frac{1}{a-j\,\Omega} + \frac{1}{a+j\,\Omega} = \frac{a+j\,\Omega + a - j\,\Omega}{a^2+\Omega^2}$$

$$= \frac{2a}{a^2+\Omega^2}$$

Example 7 Find Fourier transform of rectangular pulse function shown in figure



Solution:

$$x(t) = \pi(t) = A \quad ; \frac{-T}{2} \le t \le \frac{T}{2}$$

$$F[\pi(t)] = \int_{-\frac{T}{2}}^{\frac{T}{2}} A e^{-j\Omega t}\, dt = A\left[\frac{e^{-j\Omega t}}{-j\Omega}\right]_{-\frac{T}{2}}^{\frac{T}{2}} = \frac{A}{-j\Omega}\left[e^{-j\Omega\frac{T}{2}} - e^{j\Omega\frac{T}{2}}\right] = \frac{2A}{j\Omega}\left[\frac{e^{j\Omega\frac{T}{2}} - e^{-j\Omega\frac{T}{2}}}{2}\right] = \frac{2A}{\Omega}\sin\Omega\frac{T}{2}$$

$$= \frac{2A}{\Omega T} T \sin\Omega\frac{T}{2} = AT\frac{\sin\Omega\frac{T}{2}}{\Omega\frac{T}{2}} = AT\, \mathrm{sinc}\, \Omega\frac{T}{2}$$

Example 8 Find inverse Fourier transform $X(j\Omega) = \delta(\Omega)$

Solution:

$$\therefore F^{-1}[X(j\Omega)] = F^{-1}[\delta(\Omega)]$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\Omega) e^{j\Omega t}\, d\Omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\Omega) e^{j\Omega t}\, d\Omega = \frac{1}{2\pi}[1] \quad \because \delta(\Omega) = \begin{cases} 1 & for\ \Omega = 0 \\ 0 & for\ \Omega \ne 0 \end{cases}$$

$$F^{-1}[\delta(\Omega)] = \frac{1}{2\pi}$$

Laplace transform

It is used to transform a time domain to complex frequency domain signal(s-domain)

Two Sided Laplace transform (or) Bilateral Laplace transform

Let $x(t)$ be a continuous time signal defined for all values of t. Let $X(S)$ be Laplace transform of $x(t)$.

$$L\{x(t)\} = X(S) = \int_{-\infty}^{\infty} x(t)e^{-St}\, dt$$

One sided Laplace transform (or) Unilateral Laplace transform

Let $x(t)$ be a continuous time signal defined for $t \ge 0$ (ie If $x(t)$ is causal) then,

$$L\{x(t)\} = X(S) = \int_{0}^{\infty} x(t)e^{-St}\, dt$$

Inverse Laplace transform

The S-domain signal $X(S)$ can be transformed to time domain signal x(t) by using inverse Laplace transform.

The inverse Laplace transform of X(S) is defined as,

$$L^{-1}\{X(s)\} = x(t) = \frac{1}{2\pi j}\int_{s=\sigma-j\Omega}^{s=\sigma+j\Omega} X(S)e^{st}\,ds$$

Existence of Laplace transform

The necessary and sufficient conditions for the existence of Laplace transform are

- $x(t)$ should be continuous in the given closed interval
- $x(t)e^{-\sigma t}$ must be absolutely intergrable
  $i.e., X(S)$ *exists only if* $\int_{-\infty}^{\infty}|x(t)e^{-\sigma t}|dt < \infty$

Example 9 Find unilateral Laplace transform for the following signals

i) $\boldsymbol{x(t) = \delta(t)}$

$$X(S) = \int_0^{\infty} x(t)e^{-st}dt = \int_0^{\infty}\delta(t)e^{-st}dt = e^{-s(0)} = 1 \qquad \because \delta(t) = \begin{cases}1 & \text{for } t = 0\\ 0 & \text{for } t \neq 0\end{cases}$$

ii) $\boldsymbol{x(t) = u(t)}$

$$X(S) = \int_0^{\infty} x(t)e^{-st}dt = \int_0^{\infty} u(t)e^{-st}dt = \int_0^{\infty} 1e^{-st}dt = \left[\frac{e^{-st}}{-s}\right]_0^{\infty} = \frac{1}{s} \qquad \because u(t) = \begin{cases}1 & \text{for } t \geq 0\\ 0 & \text{for } t < 0\end{cases}$$

Example 10 Find Laplace transform of $x(t) = e^{at}u(t)$

Solution:

$$X(S) = L[e^{at}u(t)] = \int_0^{\infty} e^{at}e^{-st}\,dt = \int_0^{\infty} e^{-(s-a)t}\,dt = \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^{\infty} = \frac{1}{s-a}$$

Example 11 Determine initial value and final value of the following signal $X(S) = \frac{1}{s(s+2)}$

Solution:

Initial value

$$x(0) = \underset{s\to\infty}{\mathrm{Lt}}\ SX(S) = \underset{s\to\infty}{\mathrm{Lt}}\ s\frac{1}{s(s+2)} = \frac{1}{\infty} = 0$$

Final value

$$x(\infty) = \underset{s\to 0}{\mathrm{Lt}}\ SX(S) = \underset{s\to 0}{\mathrm{Lt}}\ s\frac{1}{s(s+2)} = \frac{1}{2}$$

Example 12 Find inverse Laplace Transform of $X(S) = \frac{S^2+9S+1}{S[S^2+6S+8]}$. Find ROC for $i)\ Re(s) > 0$ $ii)\ Re(s) < -4$ $\quad iii) -2 > Re(s) > -4$

Solution:

$$X(S) = \frac{S^2+9S+1}{S[S^2+6S+8]} = \frac{S^2+9S+1}{S(S+4)(S+2)} = \frac{A}{S} + \frac{B}{(S+4)} + \frac{C}{(S+2)}$$

$$S^2+9S+1 = A(S+4)(S+2) + BS(S+2) + CS(S+4)$$

at $S=0$	$S=-4$	$S=-2$
$A = \frac{1}{8}$	$B = -\frac{19}{8}$	$C = \frac{13}{4}$

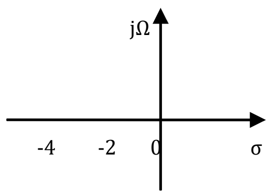
$$\therefore X(S) = \frac{\frac{1}{8}}{S} + \frac{-\frac{19}{8}}{(S+4)} + \frac{\frac{13}{4}}{(S+2)}$$

Applying inverse Laplace transform

$$x(t) = \frac{1}{8}u(t) - \frac{19}{8}e^{-4t}u(t) + \frac{13}{4}e^{-2t}u(t)$$

ROC

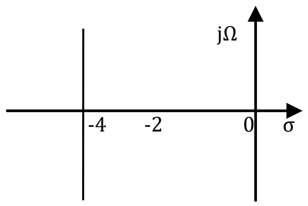
i) $\boldsymbol{Re(s) > 0}$



ROC lies right side of all poles

$$\therefore x(t) = \frac{1}{8}u(t) - \frac{19}{8}e^{-4t}u(t) + \frac{13}{4}e^{-2t}u(t)$$

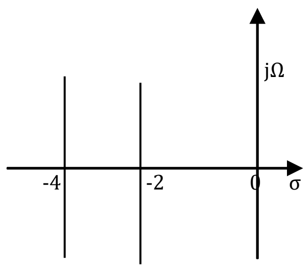
ii) $\boldsymbol{Re(s) < -4}$



ROC lies left side of all poles

$$\therefore x(t) = -\frac{1}{8}u(-t) + \frac{19}{8}e^{-4t}u(-t) - \frac{13}{4}e^{-2t}u(-t)$$

iii) $\boldsymbol{-2 > Re(s) > -4}$



ROC lies left side of poles s=-2, s=0 and right side of pole s=-4

$$\therefore x(t) = -\frac{1}{8}u(-t) \quad \frac{19}{8}e^{-4t} \quad \frac{13}{4}e^{-2t}$$

Unit 3: Linear Time Invariant-Continuous Time Systems

LTI-CT (Linear Time Invariant-Continuous Time) Systems

When continuous time system satisfies the properties of linearity and time invariant then it is called an LTI-CT (Linear Time Invariant-Continuous Time) System.

Impulse Response

When the input to a continuous time system is an unit impulse signal δ(t) then the output is called an impulse response of the system and it is denoted by h(t)

Impulse response, h(t) = H{δ(t)}

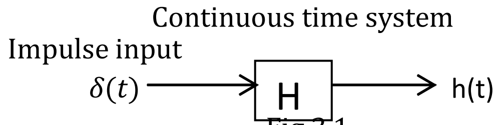


Fig 3.1

Convolution Integral

$$y(t) = \int_{-\infty}^{\infty} x(r)h(t-r)dr$$

This is called convolution integral or simply convolution. The convolution of two signal x(t) and h(t) can be represented as

$$y(t) = x(t) * h(t)$$

Systems connected in series/parallel(Block diagram representation)

System Realization

There are four types of system realization in continuous time linear time invariant systems.

They are

- Direct form I realization
- Direct form II realization
- Cascade form realization
- Parallel form realization

Direct form I realization

It is the direct implementation of differential equation or transfer function describing the system. It uses separate integrators for input and output variables. It provides direct relation between time domain and s-domain equations. In general, this form requires 2N delay elements (for both input and output signals) for a filter of order N. This form is practical for small filters.

Advantages:

- Simplicity
- Most straight forward realization

Disadvantages:

- More number of integrators are used
- Inefficient and impractical (numerically unstable) for complex design

Direct form II realization

It is the direct implementation of differential equation or transfer function describing the system. Instead of using separate integrators for integrating input and output variables separately, an intermediate variable is integrated. It provides direct relation between time domain and s-domain equations.

Advantages:

- It uses minimum number of integrators
- Straight forward realization

Disadvantages:

- It increases the possibility of arithmetic overflow for filters of high Q or resonance

Cascade form

In cascade form realization the given transfer function is expressed as a product of several transfer function and each of these transfer function is realized in direct form II and then all those realized structures are cascaded i.e., is connected in series.

Parallel form realization

The given transfer function is expressed into its partial fractions and each factor is realized in direct form II and all those realized structures are connected in parallel.

Solved Problems

Example 3.1: Find the convolution by graphical method

$$x(t) = \begin{cases} 1 \; for \;\; 0 \le t \le 2 \\ 0 \;\; otherwise \end{cases} \quad ; \quad h(t) = \begin{cases} 1 \; for \;\; 0 \le t \le 3 \\ 0 \;\; otherwise \end{cases}$$

Solution:

$$In\ general \quad x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(r)\, x_2(t-r)\, dr$$

$$Similarly\ h(t) * x(t) = \int_{-\infty}^{\infty} h(r)\, x(t-r)\, dr$$

Replacing t by r in $x(t)$and $h(t)$

$$x(r) = \begin{cases} 1 \; for \;\; 0 \le r \le 2 \\ 0 \;\; otherwise \end{cases} \quad ; \quad h(r) = \begin{cases} 1 \; for \;\; 0 \le r \le 3 \\ 0 \;\; otherwise \end{cases}$$

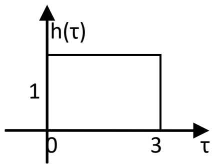


Fig 3.21

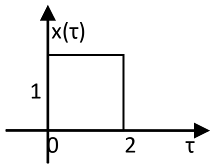


Fig 3.22

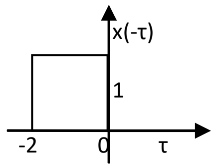


Fig 3.23

Case (i) $t < 0$

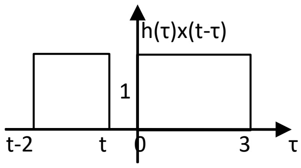


Fig 3.24

Since overlap is absent between $h(r)$ and $x(-r+t)$

$$\therefore y(t) = h(t) * x(t) = 0$$

Case (ii) $0 \leq t < 2$

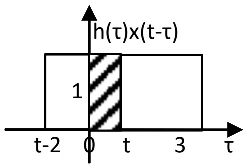


Fig 3.25

Since overlap is present

$$\therefore y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(r)\, x(t-r)\, dr = \int_{0}^{t} (1)(1)\, dr = [r]_0^t = t$$

Case (iii) $2 \leq t < 3$

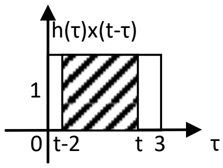


Fig 3.26

Since overlap is present

$$\therefore y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(r)\, x(t-r)\, dr = \int_{t-2}^{t} (1)(1)\, dr = [r]_{t-2}^{t} = 2$$

Case (iv) $3 \leq t < 5$

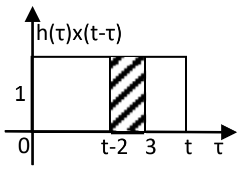


Fig 3.27

Since overlap is present

$$\therefore y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(r)\, x(t-r)\, dr = \int_{t-2}^{3} (1)(1)\, dr = [r]_{t-2}^{3}$$

$$= 5 - t$$

Case (v) $t > 5$

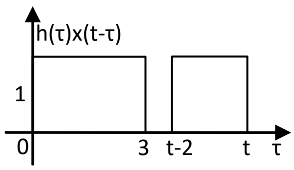


Fig 3.28

Since overlap is absent

$$\therefore y(t) = h(t) * x(t) = 0$$

$$\therefore y(t) = h(t) * x(t) = \begin{cases} 0 & for \quad t < 0 \\ t & for \quad 0 \le t < 2 \\ 2 & for \quad 2 \le t < 3 \\ 5 - t & for \quad 3 \le t < 5 \\ 0 & for \quad t \ge 5 \end{cases}$$

Example 3.2: Find impulse response of the following equation

$$\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = x(t)$$

Solution:

$$\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = x(t)$$

Assume all the initial conditions are zero

Applying Laplace transform of the given equation

$$S^2Y(S) + 5SY(S) + 6Y(S) = X(S)$$

$$Y(S)(S^2 + 5S + 6) = X(S)$$

Transfer function $H(S) = \frac{Y(S)}{X(S)} = \frac{1}{(S^2+5S+6)}$

$$H(S) = Y(S) = \frac{1}{S^2 + 5S + 6} \qquad (\because For\ impulse\ input\ x(t) = \delta(t) => X(S) = 1)$$

$$H(S) = \frac{1}{(S+3)(S+2)} = \frac{A}{S+3} + \frac{B}{S+2}$$

$$1 = A(S+2) + B(S+3)$$

at $S = -3$ | $S = -2$

$A = -1$ | $B = 1$

$$\therefore H(S) = -\frac{1}{S+3} + \frac{1}{S+2}$$

Applying Inverse Laplace transform

$$\boldsymbol{h(t) = -e^{-3t}u(t) + e^{-2t}u(t)}$$

Example 3.3: Using Laplace transform solve differential equation

$$\frac{d^2y(t)}{dt^2} + y(t) = x(t)$$

Where $y'(0) = 2; y(0) = 1; input\ x(t) = Cos2t$

Solution:

$$\frac{d^2y(t)}{dt^2} + y(t) = x(t)$$

Applying Laplace transform

$$S^2Y(S) - Sy(0) - y'(0) + Y(S) = X(S)$$

$$S^2Y(S) - S - 2 + Y(S) = \frac{S}{S^2 + 4}$$

$$\therefore Y(S)(S^2 + 1) = \frac{S}{S^2 + 4} + S + 2$$

$$Y(S) = \frac{S}{(S^2+4)(S^2+1)} + \frac{S}{(S^2+1)} + \frac{2}{(S^2+1)}$$

$$Let\ \frac{S}{(S^2+4)(S^2+1)} = \frac{AS+B}{(S^2+4)} + \frac{CS+D}{(S^2+1)}$$

$$S = (AS+B)(S^2+1) + (CS+D)(S^2+4)$$

$$S = AS^3 + BS^2 + AS + B + CS^3 + DS^2 + 4CS + 4D$$

Comparing constant term

$0 = B + 4D$

$B = -4D$ ... (7)

Comparing coeff of S^3

$0 = A + C$

$A = -C$ ... (8)

Comparing coeff of S^2

$0 = B + D$ ... (9)

Comparing coeff of S

$1 = A + 4C$ ... (10)

Substitute eq (8) in eq (10) and eq (7) in eq (9)

$\boldsymbol{C = \frac{1}{3}, D = 0}$

Substitute value of C and D in eq (8) and eq (7)

$\boldsymbol{A = -\frac{1}{3}, B = 0}$

$$\therefore \frac{S}{(S^2+4)(S^2+1)} = \frac{-\frac{1}{3}S}{(S^2+4)} + \frac{\frac{1}{3}S}{(S^2+1)}$$

$$Y(S) = \frac{-\frac{1}{3}S}{(S^2+4)} + \frac{\frac{1}{3}S}{(S^2+1)} + \frac{S}{(S^2+1)} + \frac{2}{(S^2+1)}$$

$$Y(S) = \frac{-\frac{1}{3}S}{(S^2+4)} + \frac{\frac{4}{3}S}{(S^2+1)} + \frac{2}{(S^2+1)}$$

Taking Inverse Laplace transform

$$\boldsymbol{y(t) = -\frac{1}{3}cos2t\,u(t) + \frac{4}{3}cost\,u(t) + 2sint\,u(t)}$$

Example 3.4: Find step response of the circuit shown in Fig 3.30

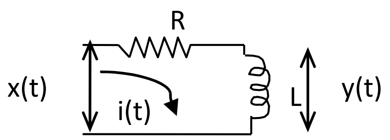


Fig 3.30

Solution:

Applying KVL to the circuit shown in Fig 3.30

$$x(t) = Ri(t) + L\frac{di(t)}{dt} \qquad y(t) = L\frac{di(t)}{dt}$$

Applying Laplace transform

$$X(S) = RI(S) + LSI(S) \qquad Y(S) = LSI(S)$$

$$X(S) = [R + LS]I(S)$$

$$I(S) = \frac{X(S)}{[R + LS]}$$

$$Y(S) = LS\frac{X(S)}{[R + LS]}$$

For Step response $x(t) = u(t) \;=>\; X(S) = \frac{1}{S}$

$$Y(S) = LS\frac{\frac{1}{S}}{R + LS} = \frac{L}{LS + R} = \frac{1}{S + \frac{R}{L}}$$

Applying Inverse Laplace transform

$$\boldsymbol{y(t) = e^{-\frac{R}{L}t}u(t)}$$

Example 3.5: Solve the differential equation using Fourier transform

$$\frac{d^2y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 8y(t) = 2x(t)$$

(i) Find the impulse response of the system

(ii) What is the response of the system if $x(t) = te^{-2t}u(t)$

Solution:

$$\frac{d^2y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 8y(t) = 2x(t)$$

Applying Fourier transform

$$(j\Omega)^2Y(j\Omega) + 6j\Omega Y(j\Omega) + 8Y(j\Omega) = 2X(j\Omega)$$

$$Y(j\Omega)[(j\Omega)^2 + 6j\Omega + 8] = 2X(j\Omega)$$

$$H(j\Omega) = \frac{Y(j\Omega)}{X(j\Omega)} = \frac{2}{[(j\Omega)^2 + 6j\Omega + 8]}$$

(i) Impulse response $x(t) = \delta(t) \;=>\; X(j\Omega) = 1$

$$\therefore H(j\Omega) = Y(j\Omega) = \frac{2}{[(j\Omega)^2 + 6j\Omega + 8]} = \frac{A}{j\Omega + 4} + \frac{B}{j\Omega + 2}$$

$$2 = A(j\Omega + 2) + B(j\Omega + 4)$$

at $j\Omega = -4$ | $j\Omega = -2$

$A = -1$ | $B = 1$

$$H(j\Omega) = \frac{-1}{j\Omega + 4} + \frac{1}{j\Omega + 2}$$

Applying Inverse Fourier Transform

$$\boldsymbol{h(t) = -e^{-4t}u(t) + e^{-2t}u(t)}$$

(ii) $x(t) = te^{-2t}u(t)$

$$X(j\Omega) = \frac{1}{(j\Omega + 2)^2}$$

$$\frac{Y(j\Omega)}{X(j\Omega)} = \frac{2}{[(j\Omega)^2 + 6j\Omega + 8]}$$

$$\therefore Y(j\Omega) = \frac{2}{[(j\Omega)^2 + 6j\Omega + 8]}\cdot\frac{1}{(j\Omega + 2)^2} = \frac{2}{(j\Omega + 4)(j\Omega + 2)^3}$$

$$= \frac{A}{j\Omega + 4} + \frac{B}{j\Omega + 2} + \frac{C}{(j\Omega + 2)^2} + \frac{D}{(j\Omega + 2)^3}$$

at $j\Omega = -4$

$$A = \frac{2}{(j\Omega+4)(j\Omega+2)^3}(j\Omega+4)\Bigg|_{j\Omega=-4}$$

$$A = -\frac{1}{4}$$

$j\Omega = -2$

$$B = \frac{1}{2!}\frac{d^2\left[\frac{2}{(j\Omega+4)(j\Omega+2)^3}(j\Omega+2)^3\right]}{d(j\Omega)^2}\Bigg|_{j\Omega=-2}$$

$$B = \frac{1}{4}$$

$j\Omega = -2$

$$C = \frac{d\left[\frac{2}{(j\Omega+4)(j\Omega+2)^3}(j\Omega+2)^3\right]}{d(j\Omega)}\Bigg|_{j\Omega=-2}$$

$$C = -\frac{1}{2}$$

$j\Omega = -2$

$$D = \frac{2}{(j\Omega+4)(j\Omega+2)^3}(j\Omega+2)^3\Bigg|_{j\Omega=-2}$$

$$D = 1$$

$$\therefore \mathrm{Y}(j\Omega) = \frac{-\frac{1}{4}}{j\Omega+4} + \frac{\frac{1}{4}}{j\Omega+2} + \frac{-\frac{1}{2}}{(j\Omega+2)^2} + \frac{1}{(j\Omega+2)^3}$$

Applying Inverse Fourier Transform

$$\boldsymbol{y(t) = -\frac{1}{4}e^{-4t}u(t) + \frac{1}{4}e^{-2t}\,u(t) - \frac{1}{2}e^{-2t}\,tu(t) + \frac{1}{2}e^{-2t}t^2u(t)}$$

Example 3.6: Find the direct form II structure of

$$H(S) = \frac{5S^3 - 4S^2 + 11S - 2}{(S-\frac{1}{4})\,(S^2 - S + \frac{1}{2})}$$

Solution:

$$H(S) = \frac{5S^3 - 4S^2 + 11S - 2}{(S-\frac{1}{4})\,(S^2 - S + \frac{1}{2})} = \frac{5S^3 - 4S^2 + 11S - 2}{S^3 - \frac{S^2}{4} - S^2 + \frac{S}{4} + \frac{S}{2} - \frac{1}{8}}$$

$$H(S) = \frac{5S^3 - 4S^2 + 11S - 2}{S^3 - \frac{5S^2}{4} + \frac{3S}{4} - \frac{1}{8}} = \frac{5 - \frac{4}{S} + \frac{11}{S^2} - \frac{2}{S^3}}{1 - \frac{5}{4S} + \frac{3}{4S^2} - \frac{1}{8S^3}}$$

Direct form II structure

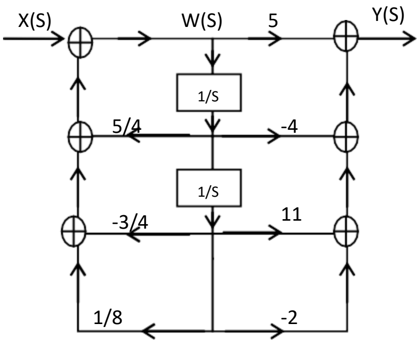


Fig 3.39

Example 3.7: Realize the system with following differential equation in direct form I

$$\frac{d^3y(t)}{dt^3} + 3\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 7y(t) = 2\frac{d^2x(t)}{dt^2} + 0.4\frac{dx(t)}{dt} + 0.5x(t)$$

Solution:

$$\frac{d^3y(t)}{dt^3} + 3\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 7y(t) = 2\frac{d^2x(t)}{dt^2} + 0.4\frac{dx(t)}{dt} + 0.5x(t)$$

Taking Laplace transform

$$S^3Y(S) + 3S^2Y(S) + 5SY(S) + 7Y(S) = 2S^2X(S) + 0.4SX(S) + 0.5X(S)$$

Dividing both the side by S^3

$$Y(S) + \frac{3}{S}Y(S) + \frac{5}{S^2}Y(S) + \frac{7}{S^3}Y(S) = \frac{2}{S}X(S) + \frac{0.4}{S^2}X(S) + \frac{0.5}{S^3}X(S)$$

$$Y(S) = \frac{2}{S}X(S) + \frac{0.4}{S^2}X(S) + \frac{0.5}{S^3}X(S) - \frac{3}{S}Y(S) - \frac{5}{S^2}Y(S) - \frac{7}{S^3}Y(S)$$

Direct form I structure

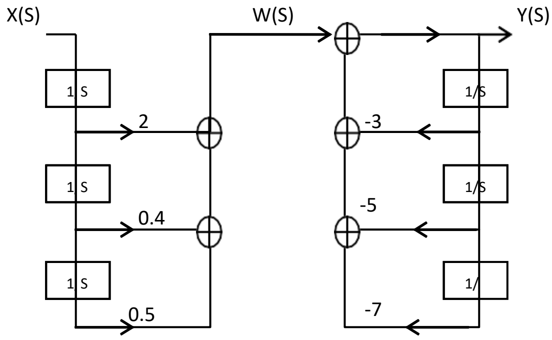


Fig 3.42

Example 3.8: Realize the system with transfer function in cascade form

$$H(S) = \frac{4(S^2 + 4S + 3)}{S^3 + 6.5S^2 + 11S + 4}$$

Solution:

$$H(S) = \frac{4(S^2 + 4S + 3)}{S^3 + 6.5S^2 + 11S + 4} = \frac{4(S + 1)(S + 3)}{(S + 0.5)(S + 2)(S + 4)} = \frac{4}{S + 0.5}\cdot\frac{S + 1}{S + 2}\cdot\frac{S + 3}{S + 4}$$

$$H_1(S)H_2(S)H_3(S) = \frac{4}{S + 0.5}\cdot\frac{S + 1}{S + 2}\cdot\frac{S + 3}{S + 4}$$

$$H_1(S) = \frac{4}{S + 0.5} = \frac{4/S}{1 + 0.5/S}$$

$$\frac{Y_1(S)}{W_1(S)} = \frac{4}{S}$$

$$\frac{W_1(S)}{X_1(S)} = \frac{1}{1 + \frac{0.5}{S}}$$

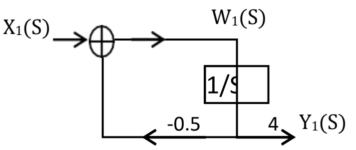


Fig 3.54

$$H_2(S) = \frac{S + 1}{S + 2} = \frac{1 + 1/S}{1 + 2/S}$$

$$\frac{Y_2(S)}{W_2(S)} = 1 + 1/S$$

$$\frac{W_2(S)}{X_2(S)} = \frac{1}{1 + 2/S}$$

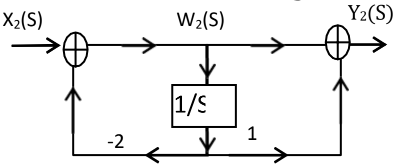


Fig 3.55

$$H_3(S) = \frac{S + 3}{S + 4} = \frac{1 + 3/S}{1 + 4/S}$$

$$\frac{Y_3(S)}{W_3(S)} = 1 + 3/S$$

$$\frac{W_3(S)}{X_3(S)} = \frac{1}{1 + 4/S}$$

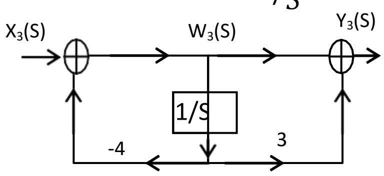


Fig 3.56

Cascade form:

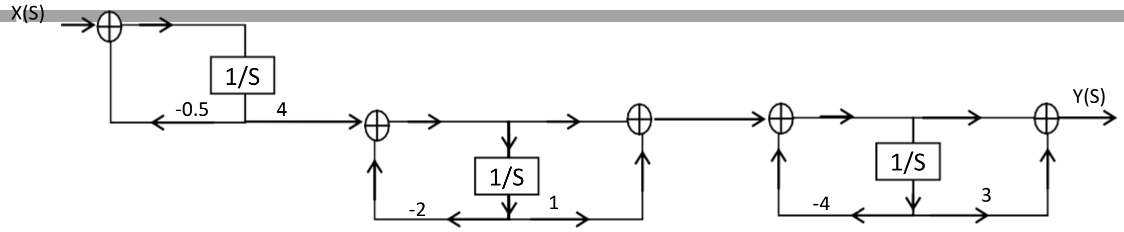


Fig 3.57

Example 3.9: Realize the following system in parallel form

$$H(S) = \frac{S(S+2)}{(S+1)(S+3)(S+4)}$$

Solution:

$$H(S) = \frac{S(S+2)}{(S+1)(S+3)(S+4)} = \frac{A}{S+1} + \frac{B}{S+3} + \frac{C}{S+4}$$

$$S(S+2) = A(S+3)(S+4) + B(S+1)(S+4) + C(S+1)(S+3)$$

Let S $= -1$

$-1(1) = A(2)(3)$

$\boldsymbol{A = -\frac{1}{6}}$

Let S $= -3$

$-3(-1) = B(-2)(1)$

$\boldsymbol{B = -\frac{3}{2}}$

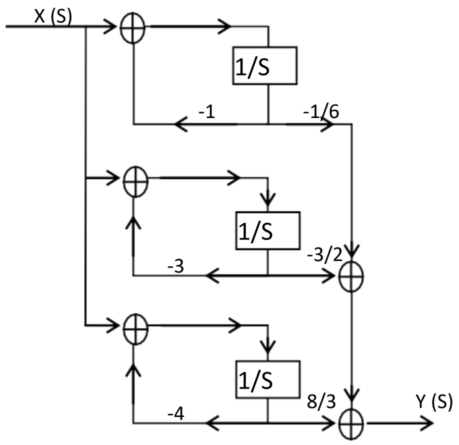
Let S $= -4$

$-4(-2) = C(-3)(-1)$

$\boldsymbol{C = \frac{8}{3}}$

$$\therefore H(S) = \frac{-\frac{1}{6}}{S+1} + \frac{-\frac{3}{2}}{S+3} + \frac{\frac{8}{3}}{S+4}$$

Parallel form structure



Unit 4: Analysis of Discrete Time Signals

Sampling of CT signals

Sampling theorem (or) uniform sampling theorem (or) Low pass sampling theorem

It is one of useful theorem that applies to digital communication systems.

Sampling theorem states that “A band limited signal $\boldsymbol{x(t)}$ with $\boldsymbol{X}(m) = \boldsymbol{0}$ for $|m| \geq m\boldsymbol{m}$ can be represented into and uniquely determined from its samples $\boldsymbol{x(nT)}$ if the sampling frequency $\boldsymbol{f_s} \geq \boldsymbol{2f_m}$, where $\boldsymbol{f_m}$ is the frequency component present in it”.

(i.e) for signal recovery, the sampling frequency must be at least twice the highest frequency present in the signal.

Proof:

Sampling Operation:

δT(t) is impulse train

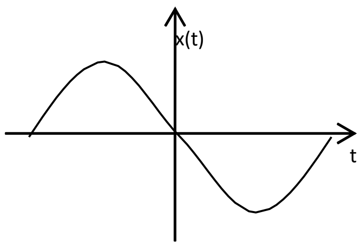


Fig 4.1

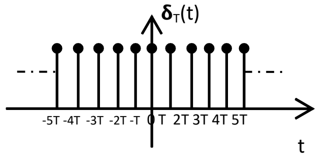


Fig 4.2

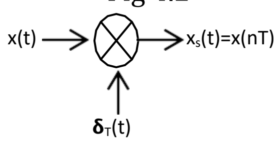


Fig 4.3

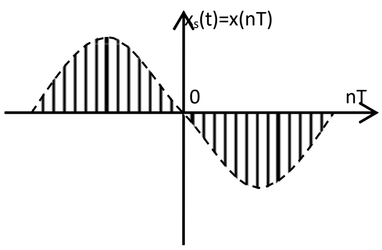


Fig 4.4

Analog signal $x(t)$ is input signal as shown in Fig 4.1, $\delta_T(t)$ is the train of impulse shown in Fig 4.2
Sampled signal $x_s(t)$ is the product of signal $x(t)$ and impulse train $\delta_T(t)$ as shown in Fig 4.2

$$\therefore x_s(t) = x(t).\,\delta_T(t)$$

$$we\ know\ \delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) = \frac{1}{T}\sum_{n=-\infty}^{\infty} e^{jn\,\omega_s t}$$

$$\therefore x_s(t) = x(t).\frac{1}{T}\sum_{n=-\infty}^{\infty} e^{jn\,\omega_s t}$$

Applying Fourier transform on both sides

$$X_s(\omega) = \frac{1}{T}\sum_{n=-\infty}^{\infty} F[x(t)e^{jn\,\omega_s t}]$$

$$X_s(\omega) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X(\omega - n\omega_s)$$

$$where\ \omega_s = 2\pi f_s = \frac{2\pi}{T}$$

$$\therefore X_s(\omega) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X(\omega - \frac{2\pi n}{T})$$

$$(or)$$

$$X_s(f) = f_s \sum_{n=-\infty}^{\infty} X(f - nf_s) \qquad where\ f_s = \frac{1}{T}$$

Where $X(\omega)$ or $X(f)$ $is\ Spectrum\ of\ input\ signal.$

Where $X_s(\omega)$ or $X_s(f)$ $is\ Specturm\ of\ sampled\ signal.$

Spectrum of continuous time signal x(t) with maximum frequency ω_m is shown in Fig 4.5.

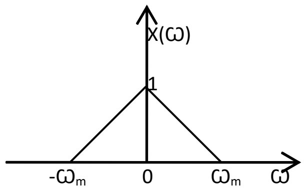


Fig 4.5 Spectrum of x(t)

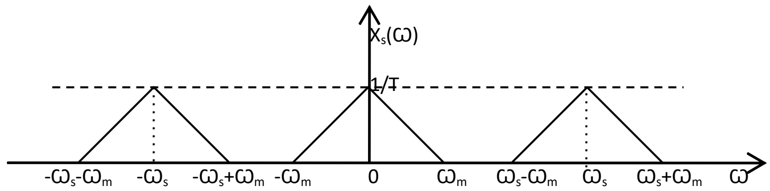


Fig 4.6 Spectrum of $x_s(t)$ when $m_s - m_m > m_m$

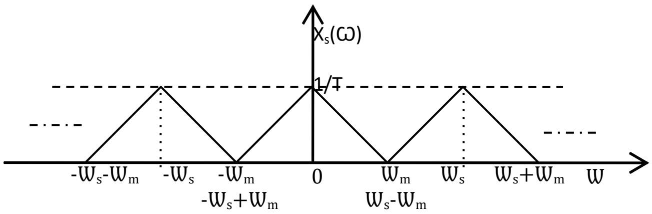


Fig 4.7 Spectrum of $x_s(t)$ when $m_s - m_m = m_m$

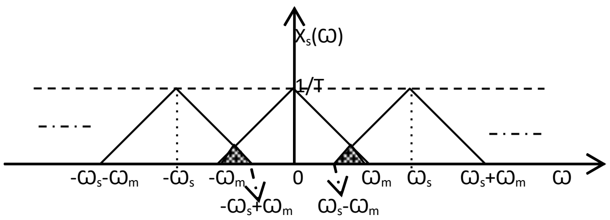


Fig 4.8 Spectrum of $x_s(t)$ when $m_s - m_m < m_m$

From the plot of $X_s(\omega)$ (Fig 4.6, Fig 4.7, Fig 4.8),

For $m_s > 2m_m$

The spectral replicates have a larger separation between them known as guard band which makes process of filtering much easier and effective. Even a non-ideal filter which does not have a sharp cut off can also be used.

For $m_s = \mathbf{2}m_m$

There is no separation between the spectral replicates so no guard band exists and $X(\omega)$ can be obtained from $X_s(\omega)$ by using only an ideal low pass filter (LPF) with sharp cutoff.

For $m_s < 2m_m$

The low frequency component in $X_s(\omega)$ overlap on high frequency components of $X(\omega)$ so that there is presence of distortion and $X(\omega)$ cannot be recovered from $X_s(\omega)$ by using any filter. This distortion is called aliasing.

So we can conclude that the frequency spectrum of $X_s(\omega)$ is not overlapped for $m_s - m_m \geq m_m$, therefore the Original signal can be recovered from the sampled signal.

For $m_s - m_m < m_m$, the frequency spectrum will overlap and hence the original signal cannot be recovered from the sampled signal.

∴ For signal recovery,

$$\omega_s - \omega_m \geq \omega_m \,(i.e)\;\; m_s \geq \mathbf{2}m_m$$

$$(or)$$

$$\boldsymbol{f_s \geq 2f_m}$$

i.e., Aliasing can be avoided if $f_s \geq 2f_m$

Aliasing effect (or) fold over effect

It is defined as the phenomenon in which a high frequency component in the frequency spectrum of signal takes identity of a lower frequency component in the spectrum of the sampled signal.

When $f_s < 2f_m$, (i.e) when signal is under sampled, the individual terms in equation $X_s(\omega) = \frac{1}{T}\sum_{n=-\infty}^{\infty} x(\omega - n\omega_s)$ get overlap. This process of spectral overlap is called frequency folding effect.

Occurrence of aliasing

Aliasing Occurs if

i) The signal is not band-Limited to a finite range.

ii) The sampling rate is too low.

To Avoid Aliasing

i) $x(t)$ should be strictly band limited.

It can be ensured by using anti-aliasing filter before the sampler.

ii) f_s should be greater than $2f_m$.

Nyquist rate

It is the theoretical minimum sampling rate at which a signal can be sampled and still be reconstructed from its samples without any distortion

$$Nyquist\ rate\ f_N = 2f_m.Hz$$

Data Reconstruction or Interpolation

The process of obtaining analog signal $x(t)$ from the sampled signal $x_s(t)$ is called data reconstruction or interpolation.

$$we\ know\ x_s(t) = x(t).\delta_T(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t-nT)$$

$$\delta(t-nT)\ exist\ \ only\ at\ t = nT$$

$$\therefore x_s(t) = x(nt) \sum_{n=-\infty}^{\infty} \delta(t-nT)$$

The reconstruction filter, which is assumed to be linear and time invariant, has unit impulse response $h(t)$.

The reconstruction filter, output $y(t)$ is given by convolution of $x_s(t)$ and $h(t)$.

$$\therefore y(t) = x_s(t) * h(t) = \int_{-\infty}^{\infty} x(nT) \sum_{n=-\infty}^{\infty} \delta(r-nT).\ h(t-r)dr$$

$$= \sum_{n=-\infty}^{\infty} x(nT) \int_{-\infty}^{\infty} \delta(r-nT)\ h(t-r)dr$$

$$\delta(r-nT) exist\ only\ at\ r = nT$$

$$\delta(r-nT) = 1\ at\ r = nT$$

$$\therefore y(t) = \sum_{n=-\infty}^{\infty} x(nT)\ h(t-nT)$$

Ideal Reconstruction filter

The sampled signal $x_s(t)$ is passed through an ideal LPF (Fig 4.9) with bandwidth greater than f_m and a pass band amplitude response of T, then the filter output is $x(t)$.

Transfer function of ideal reconstruction filter is

$$H(f) = T\ ;\ |f| < 0.5 f_s$$

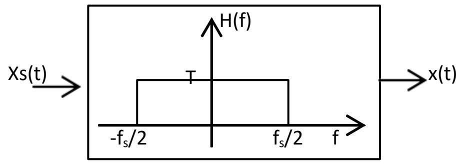


Fig 4.9

The impulse response of ideal reconstruction filter is

$$h(t) = \int_{-\frac{f_s}{2}}^{\frac{f_s}{2}} T\ e^{j\omega t}\ df = \int_{-\frac{f_s}{2}}^{\frac{f_s}{2}} T\ e^{j2\pi ft}\ df = T\left[\frac{e^{j2\pi ft}}{j2\pi t}\right]_{-\frac{f_s}{2}}^{\frac{f_s}{2}} = \frac{T}{j2\pi t}\left[e^{j2\pi\frac{f_s}{2}t} - e^{-j2\pi\frac{f_s}{2}t}\right]$$

$$= \frac{1}{f_s\pi t}\left[\frac{e^{j2\pi\frac{f_s}{2}t} - e^{-j2\pi\frac{f_s}{2}t}}{2j}\right] = \frac{1}{\pi f_s t}\sin \pi f_s t = \sin c\ \pi f_s t$$

$$\therefore h(t-nT) = \sin c\ \pi f_s(t-nT)\ \ \ldots\ldots\ldots (1)$$

$$y(t) = \sum_{n=-\infty}^{\infty} x(nT) h(t-nT)$$

Substitute equation (1) in above equation

$$\therefore y(t) = \sum_{n=-\infty}^{\infty} x(nT)\ \sin c\ \pi f_s(t-nT) = \sum_{n=-\infty}^{\infty} x(nT)\ \sin c\ \pi\left(\frac{t}{T}-n\right)$$

$$[\because f_s = \frac{1}{T}]$$

Example 4.1: Determine Nyquist rate and Nyquist interval corresponding to each of the following signals

$$x(t) = \cos 1000\pi t + \cos 3000\pi t + \sin 4000\pi t$$

$\omega_m = 4000\pi \Rightarrow 2\pi f_m = 4000\pi \Rightarrow f_m = 2000Hz$

$Nyquist\ rate = 2f_m = 4000\ Hz$

$Nyquist\ interval = \dfrac{1}{2f_m} = \dfrac{1}{4000} = 0.25\ mS$

Discrete time Fourier Transform

$$F[x(n)] = X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

4.2.2 Inverse Discrete Time Fourier Transform

$$x(n) = F^{-1}[X(e^{j\omega})] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega\ , for\ n = -\infty\ to\ \infty$$

Example 4.9: Find Fourier transform of the following

i) $x(n) = \delta(n)$

ii) $x(n) = u(n)$

iii) $x(n) = a^n u(n)$

Solution:

i) $x(n) = \delta(n)$

$$X(e^{j\omega}) = F\{\delta(n)\} = \sum_{n=-\infty}^{\infty} \delta(n)\, e^{-j\omega n} = e^0 = 1$$

$$\delta(n) = \begin{cases} 0, & n \neq 0 \\ 1, & n = 0 \end{cases}$$

ii) $x(n) = u(n)$

$$X(e^{j\omega}) = F\{u(n)\} = \sum_{n=-\infty}^{\infty} u(n)\, e^{-j\omega n} = \sum_{n=0}^{\infty} e^{-j\omega n}$$

$$= 1 + e^{-j\omega} + e^{-2j\omega} + \cdots\infty = \frac{1}{1-e^{-j\omega}}$$

$$u(n) = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases}$$

$$1 + x + x^2 + \cdots = \frac{1}{1-x}$$

iii) $x(n) = a^n u(n)$

It is Right handed exponential signal

$$X(e^{j\omega}) = F\{a^n u(n)\} = \sum_{n=-\infty}^{\infty} a^n u(n) e^{-j\omega n} = \sum_{n=0}^{\infty} a^n e^{-j\omega n}$$

$$= 1 + ae^{-j\omega} + (ae^{-j\omega})^2 + \cdots + \infty = \frac{1}{1 - ae^{-j\omega}}$$

Example 4.11: Obtain DTFT of rectangular pulse

$$x(n) = \begin{cases} A, & 0 \le n \le L-1 \\ 0, & otherwise \end{cases}$$

Solution:

$$X(e^{j\omega}) = \sum_{n=0}^{L-1} Ae^{-j\omega n} = A\left[\frac{1 - e^{-j\omega L}}{1 - e^{-j\omega}}\right] \qquad \sum_{n=0}^{N-1} x^n = \frac{1 - x^N}{1 - x}$$

$$X(e^{j\omega}) = A\left[\frac{\left(e^{\frac{j\omega L}{2}} - e^{\frac{-j\omega L}{2}}\right) e^{\frac{-j\omega L}{2}}}{\left(e^{\frac{j\omega}{2}} - e^{-\frac{j\omega}{2}}\right) e^{\frac{-j\omega}{2}}}\right] = A\left[\frac{2j \sin\frac{\omega L}{2}}{2j \sin\frac{\omega}{2}}\right] e^{-\frac{j\omega (L-1)}{2}} = Ae^{-\frac{j\omega (L-1)}{2}} \left[\frac{\sin\frac{\omega L}{2}}{\sin\frac{\omega}{2}}\right]$$

Example 4.16: Find DTFT of $x(n) = \sin(n\,\theta)u(n)$

Solution:

$$x(n) = \sin(n\,\theta)u(n)$$

$$X(e^{j\omega}) = \sum_{n=0}^{\infty} \sin(n\,\theta) e^{-j\omega n} = \sum_{n=0}^{\infty} \left(\frac{e^{j\theta n} - e^{-j\theta n}}{2j}\right) e^{-j\omega n} = \frac{1}{2j}\left(\sum_{n=0}^{\infty} e^{j(\theta-\omega)n} - \sum_{n=0}^{\infty} e^{-j(\theta+\omega)n}\right)$$

$$= \frac{1}{2j}\left(\frac{1}{1 - e^{j(\theta-\omega)}} - \frac{1}{1 - e^{-j(\theta+\omega)}}\right) = \frac{1}{2j}\left(\frac{1 - e^{-j(\theta+\omega)} - 1 + e^{j(\theta-\omega)}}{1 - 2e^{-j\omega}\cos\theta + e^{-2j\omega}}\right)$$

$$= \frac{1}{2j}\left(\frac{2je^{-j\omega}\sin\theta}{1 - 2e^{-j\omega}\cos\theta + e^{-2j\omega}}\right) = \frac{e^{-j\omega}\sin\theta}{1 - 2e^{-j\omega}\cos\theta + e^{-2j\omega}}$$

4.4 Z-Transform

The Z-transform of discrete time signal $x(n)$ is defined as

$$Z[x(n)] = X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n}$$

4.4.2 Inverse Z-transform

The inverse Z-transform of $X(Z)$ is defined as

$$x(n) = Z^{-1} X(Z) = \frac{1}{2\pi j} \oint_C X(Z)\, Z^{n-1} dZ$$

Example 4.21: Find Z-transform of the following

i) $x(n) = \delta(n)$

ii) $x(n) = u(n)$

iii) $x(n) = -a^n u(-n-1)$ *and find ROC*

i) $x(n) = \delta(n)$

$$\delta(n) = \begin{cases} 1 \; for \; n = 0 \\ 0 \; for \; n \neq 0 \end{cases}$$

$$Z[\delta(n)] = \sum_{n=-\infty}^{\infty} \delta(n) z^{-n} = Z^{-0} = 1$$

$$\therefore Z[\delta(n)] = 1$$

ii) $x(n) = u(n)$

$$Z[u(n)] = \sum_{n=-\infty}^{\infty} u(n) z^{-n} = \sum_{n=0}^{\infty} z^{-n} = 1 + z^{-1} + z^{-2} + z^{-3} + \cdots\cdots = 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots\cdots$$

$$= \frac{1}{1 - \frac{1}{z}} = \frac{1}{1 - z^{-1}} = \frac{Z}{Z - 1}$$

The above series convergence if $|Z^{-1}| < 1$ i.e ROC is $|Z| > 1$

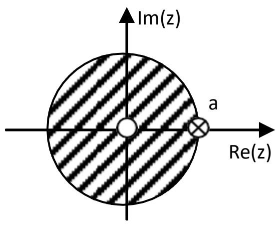
iii) $x(n) = -a^n u(-n-1)$

$$X(Z) = \sum_{n=\infty}^{\infty} x(n) z^{-n} = \sum_{n=-\infty}^{\infty} -a^n u(-n-1) z^{-n} = -\sum_{n=-\infty}^{-1} a^n z^{-n}$$

$$X(Z) = -\sum_{n=1}^{\infty} a^{-n} z^n = -[a^{-1}Z + (a^{-1}Z)^2 + (a^{-1}Z)^3 + \cdots] = -a^{-1}Z[1 + a^{-1}Z + (a^{-1}Z)^2 + \cdots]$$

$$= -[\frac{a^{-1}z}{1 - a^{-1}z}] = \frac{-a^{-1}}{-a^{-1}z}[\frac{z}{1 - az^{-1}}] = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}$$

ROC: $|a^{-1}Z| < 1 \Rightarrow |Z| < |a|$



xample 4.25: Obtain Inverse Z-Transform of

$$X(z) = \frac{1}{1 - 0.6Z^{-1} + 0.08Z^{-2}} \quad for \;\; i) \; |Z| > 0.4 \;\; ii) \; |Z| < 0.2 \;\; iii) \; 0.2 < |Z| < 0.4$$

Solution:

$$X(z) = \frac{1}{1 - 0.6Z^{-1} + 0.08Z^{-2}} = \frac{Z^2}{Z^2 - 0.6Z + 0.08}$$

$$\frac{X(Z)}{Z} = \frac{Z}{(Z - 0.2)(Z - 0.4)} = \frac{A}{Z - 0.2} + \frac{B}{Z - 0.4}$$

$$A = \frac{Z}{(Z - 0.2)(Z - 0.4)}(Z - 0.2)\Big|_{Z=0.2} = -1 \quad \Bigg| \quad B = \frac{Z}{(Z - 0.2)(Z - 0.4)}(Z - 0.4)\Big|_{Z=0.4} = 2$$

$$\therefore \frac{X(Z)}{Z} = \frac{-1}{Z - 0.2} + \frac{2}{Z - 0.4} \Rightarrow X(Z) = \frac{-Z}{Z - 0.2} + \frac{2Z}{Z - 0.4}$$

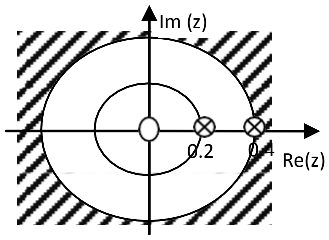
Applying inverse Z-transform

$$x(n) = -(0.2)^n u(n) + 2(0.4)^n u(n)$$

ROC: $|Z| > 0.4$

ROC lies outside of all poles. So both the terms are causal

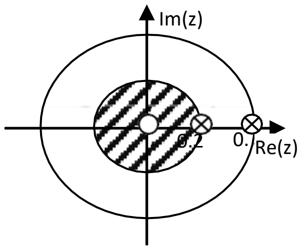
$\therefore x(n) = -(0.2)^n u(n) + 2(0.4)^n u(n)$



ROC : $|Z| < 0.2$

ROC lies inside of all poles. So both the terms are non-causal

$\therefore x(n) = (0.2)^n u(-n-1) - 2(0.4)^n u(-n-1)$

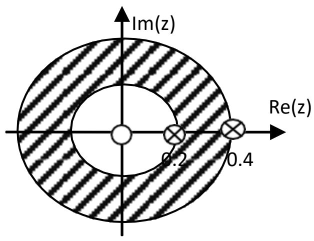


ROC :

$0.5 < |z| < 1$

ROC lies inside of pole Z=0.4 and lies outside of pole Z=0.2. So the term with pole Z=0.4 is non-causal and the term with pole Z=0.2 is causal

$\therefore x(n) = -(0.2)^n u(n) - 2(0.4)^n u(-n-1)$



Example 4.29: Find the inverse Z transform of $X(Z) = \frac{Z^3}{(Z+1)(Z-1)^2}$ using Cauchy residue method.

Solution:

$$X(Z) = \frac{Z^3}{(Z+1)(Z-1)^2}$$

$$x(n) = Residue\ of\ X(Z)\ Z^{n-1} at\ pole\ (Z=-1)$$
$$+ Residue\ of X(Z)\ Z^{n-1} at\ pole\ (Z=1)\ wit\square\ multiplicity\ 2$$

$$x(n) = \frac{Z^3(Z+1)}{(Z+1)(Z-1)^2} Z^{n-1}\Big|_{Z=-1} + \frac{d}{dZ}\left[\frac{Z^3(Z-1)^2}{(Z+1)(Z-1)^2}\right] Z^{n-1}\Big|_{Z=1}$$

$$= \left(\frac{1}{4}\right)(-1)^{n-1} + \frac{2(n+2)-1}{4} = \left(-\frac{1}{4}\right)(-1)^{n-1} u(n) + \frac{2n+3}{4} u(n)$$

$x(0) = 1 \quad x(1) = 1 \quad x(2) = 2 \quad x(3) = 2$

$\therefore x(n) = \{1,1,2,2,\ldots\}$

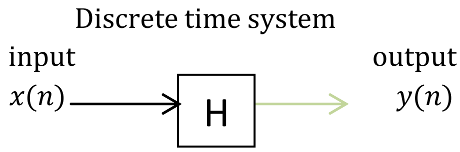
UNIT 5: Linear Time Invariant –Discrete Time Systems

Linear Time Invariant Discrete Time System (LTI-DT System)

When a discrete time system satisfies the properties of linearity and time invariance, then it is called an LTI System.

Discrete time system

A discrete time system is a device that operates on a discrete time signal to produce another discrete time signal called the output or response of the system.



The input signal x(n) is transformed to output signal y(n) through the above system

Impulse Response

When the input to a discrete time system is a unit impulse $\delta(n)$ then the output is called an impulse response of the system and is denoted by $h(n)$

$$\therefore \text{Impulse response } h(n) = \mathrm{H}\{\delta(n)\}$$

$$\delta(n) \rightarrow \boxed{H} \rightarrow h(n)$$

Impulse response of interconnected systems

Parallel connections of discrete time systems (Distributive property)

Consider two LTI systems with impulse response $h_1(n)$ and $h_2(n)$ connected in parallel as shown in Fig 5.1

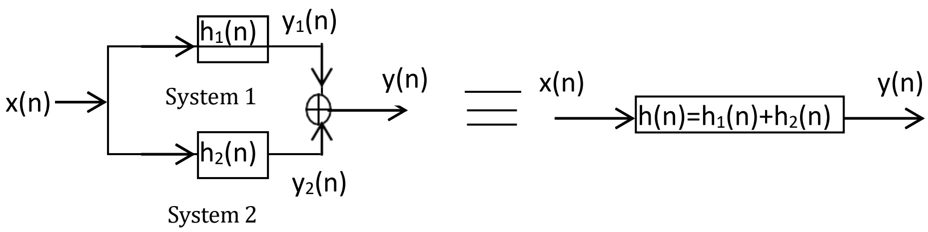


Fig 5.1 Parallel connections of discrete time systems

Cascade connection of discrete time systems (Associative property)

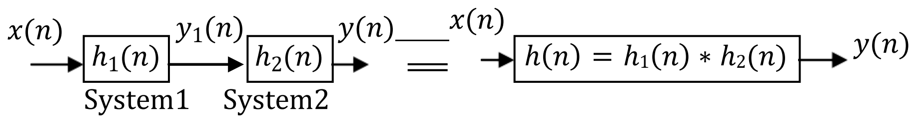


Fig 5.2

Let us consider two systems with impulse $h_1(n)$ and $h_2(n)$ connected in cascade as shown in Fig 5.2

Block diagram representation (System connected in series/parallel)

System Realization

There are four types of system realization in discrete time linear time invariant systems. They are

- Direct form I realization
- Direct form II realization
- Cascade form realization
- Parallel form realization

Direct form I realization

It is the direct implementation of transfer function describing the system. It uses separate unit delay element for input and output variables. It provides direct relation between time domain and Z-domain equations. This form is practical for small filters.

Advantages:

- Simplicity
- Most straight forward realization

Disadvantages:

- More number of unit delay elements are used
- Inefficient and impractical for complex design

Consider a system with system function

$$H(Z) = \frac{Y(Z)}{X(Z)} = \frac{b_0 + b_1Z^{-1} + b_2Z^{-2}}{1 + a_1Z^{-1} + a_2Z^{-2}}$$

$$Y(Z) + a_1Z^{-1}Y(Z) + a_2Z^{-2}Y(Z) = b_0X(Z) + b_1Z^{-1}X(Z) + b_2Z^{-2}X(Z)$$

Direct form – I realization of $H(Z)$

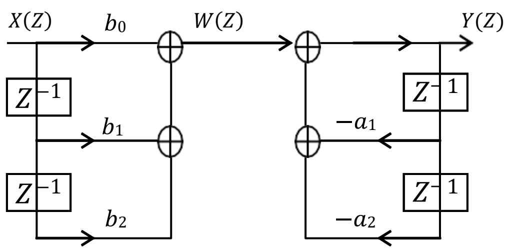


Fig 5.3 Direct form – I realization

Direct form II realization

It is the direct implementation of transfer function describing the system. Instead of using separate unit delay elements for input and output variables separately, an intermediate variable is unit delay element. It provides direct relation between time domain and z-domain equations.

Advantages:

It uses minimum number of unit delay element

Straight forward realization

Consider a system with system function

$$H(Z) = \frac{Y(Z)}{X(Z)} = \frac{b_0 + b_1Z^{-1} + b_2Z^{-2}}{1 + a_1Z^{-1} + a_2Z^{-2}} \quad \ldots\ldots\ldots\langle 4\rangle$$

$$Let\ \frac{Y(Z)}{X(Z)} = \frac{Y(Z)}{W(Z)} \cdot \frac{W(Z)}{X(Z)}$$

$$Where\ \frac{W(Z)}{X(Z)} = \frac{1}{1 + a_1 Z^{-1} + a_2 Z^{-2}} \quad \dots \dots \dots \langle 5 \rangle$$

$$and\ \frac{Y(Z)}{W(Z)} = b_0 + b_1 Z^{-1} + b_2 Z^{-2} \dots \dots \dots \langle 6 \rangle$$

From eq ⟨5⟩ we have

$$W(Z) = X(Z) - a_1 Z^{-1} W(Z) - a_2 Z^{-2} W(Z) \dots \dots \dots \langle 7 \rangle$$

From eq ⟨6⟩ we have

$$Y(Z) = b_0 W(Z) + b_1 Z^{-1} W(Z) + b_2 Z^{-2} W(Z) \dots \dots \dots \langle 8 \rangle$$

Realization of eq ⟨7⟩

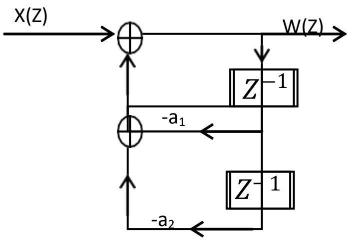


Fig 5.4

Realization of eq ⟨8⟩

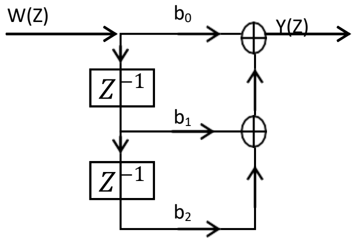


Fig 5.5

Combined form of Fig 5.4 and Fig 5.5 gives direct form II realization as shown in Fig 5.6

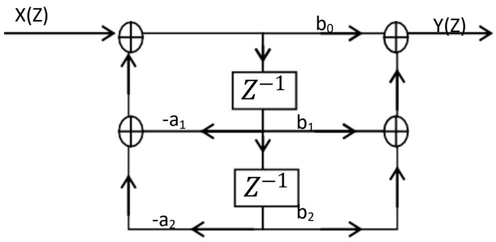


Fig 5.6 Direct form II realization

Cascade form(Series form)

In cascade form realization the given transfer function is expressed as a product of several transfer function and each of these transfer function is realized in direct form II and then all those realized structures are cascaded i.e., connected in series.

Consider a system with the following system function

$$H(Z) = \frac{(b_{k0} + b_{k1}Z^{-1} + b_{k2}Z^{-2})(b_{m0} + b_{m1}Z^{-1} + b_{m2}Z^{-2})}{(1 + a_{k1}Z^{-1} + a_{k2}Z^{-2})(1 + a_{m1}Z^{-1} + a_{m2}Z^{-2})} = H_1(Z)H_2(Z)$$

Where

$$H_1(Z) = \frac{(b_{k0} + b_{k1}Z^{-1} + b_{k2}Z^{-2})}{(1 + a_{k1}Z^{-1} + a_{k2}Z^{-2})}$$

$$H_2(Z) = \frac{(b_{m0} + b_{m1}Z^{-1} + b_{m2}Z^{-2})}{(1 + a_{m1}Z^{-1} + a_{m2}Z^{-2})}$$

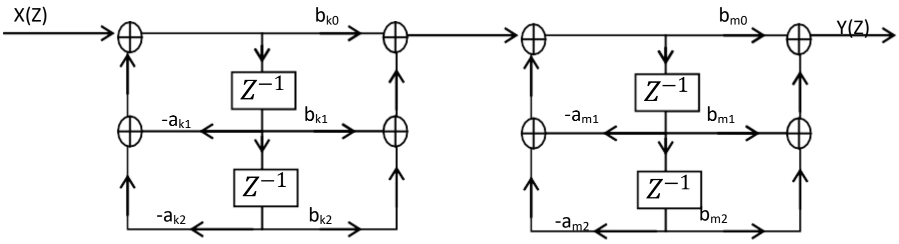


Fig 5.7 Cascade form realization

Realizing $H_1(Z)$ and $H_2(Z)$ in direct form II and the cascade form of the system function $H(Z)$ is shown in Fig 5.7

Parallel form realization

The given transfer function is expressed into its partial fractions and each factor is realized in direct form II and all those realized structures are connected in parallel as shown in Fig 5.8.

Consider a system with the following system function

$$H(Z) = c + \sum_{k=1}^{N} \frac{C_k}{1 - P_k Z^{-1}}$$

Where $\{P_k\}$ are poles of the system function

$$H(Z) = c + \frac{C_1}{1 - P_1 Z^{-1}} + \frac{C_2}{1 - P_2 Z^{-1}} + \dots\dots\dots + \frac{C_N}{1 - P_N Z^{-1}}$$

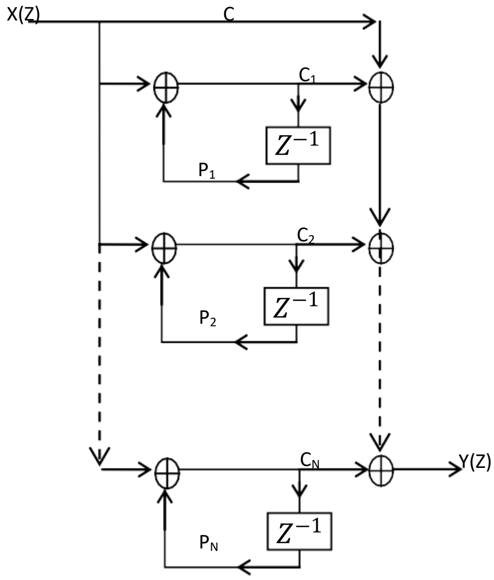


Fig 5.8 Parallel form realization

Solved Problems

Example 5.1: Determine the frequency response and impulse response

$$y(n) - \frac{1}{6}y(n-1) - \frac{1}{6}y(n-2) = x(n)$$

Solution:

$$y(n) - \frac{1}{6}y(n-1) - \frac{1}{6}y(n-2) = x(n)$$

Applying DTFT

$$Y(e^{j\omega}) - \frac{1}{6}e^{-j\omega}Y(e^{j\omega}) - \frac{1}{6}e^{-2j\omega}Y(e^{j\omega}) = X(e^{j\omega})$$

Frequency response $H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1}{1-\frac{1}{6}e^{-j\omega}-\frac{1}{6}e^{-2j\omega}} = \frac{e^{2j\omega}}{e^{2j\omega}-\frac{1}{6}e^{j\omega}-\frac{1}{6}}$

$$\frac{H(e^{j\omega})}{e^{j\omega}} = \frac{e^{j\omega}}{e^{2j\omega}-\frac{1}{6}e^{j\omega}-\frac{1}{6}} = \frac{A}{e^{j\omega}-\frac{1}{2}} + \frac{B}{e^{j\omega}+\frac{1}{3}}$$

$$e^{j\omega} = A\,(e^{j\omega}+\frac{1}{3}) + B\,(e^{j\omega}-\frac{1}{2})$$

At $e^{j\omega} = -\frac{1}{3}$

$-\frac{1}{3} = B(-\frac{1}{3}-\frac{1}{2})$, $\therefore B = \frac{2}{5}$

At $e^{j\omega} = \frac{1}{2}$

$\frac{1}{2} = A(\frac{1}{2}+\frac{1}{3})$, $\therefore A = \frac{3}{5}$

$$H(e^{j\omega}) = \frac{\frac{3}{5}e^{j\omega}}{e^{j\omega}-\frac{1}{2}} + \frac{\frac{2}{5}e^{j\omega}}{e^{j\omega}+\frac{1}{3}}$$

Applying inverse DTFT

$$h(n) = \frac{3}{5}(\frac{1}{2})^n u(n) + \frac{2}{5}(-\frac{1}{3})^n u(n)$$

Example 5.2: Find response of system using DTFT

$$h(n) = (\frac{1}{2})^n u(n) \ ; \ x(n) = (\frac{3}{4})^n u(n)$$

Solution:

$$h(n) = (\frac{1}{2})^n u(n) \ ; \ x(n) = (\frac{3}{4})^n u(n)$$

Applying DTFT

$$H(e^{j\omega}) = \frac{1}{1-\frac{1}{2}e^{-j\omega}} \quad ; \quad X(e^{j\omega}) = \frac{1}{1-\frac{3}{4}e^{-j\omega}}$$

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$

$$Y(e^{j\omega}) = \frac{1}{1-\frac{1}{2}e^{-j\omega}} \cdot \frac{1}{1-\frac{3}{4}e^{-j\omega}} = \frac{e^{j\omega}}{e^{j\omega}-\frac{1}{2}} \cdot \frac{e^{j\omega}}{e^{j\omega}-\frac{3}{4}}$$

$$\frac{Y(e^{j\omega})}{e^{j\omega}} = \frac{e^{j\omega}}{(e^{j\omega}-\frac{1}{2})(e^{j\omega}-\frac{3}{4})} = \frac{A}{e^{j\omega}-\frac{1}{2}} + \frac{B}{e^{j\omega}-\frac{3}{4}}$$

$$e^{j\omega} = A\,(e^{j\omega}-\frac{3}{4}) + B\,(e^{j\omega}-\frac{1}{2})$$

At $e^{j\omega} = \frac{1}{2}$

$\frac{1}{2} = A(\frac{1}{2}-\frac{3}{4})$, $\therefore A = -2$

At $e^{j\omega} = \frac{3}{4}$

$\frac{3}{4} = B(\frac{3}{4}-\frac{1}{2})$, $\therefore B = 3$

$$\frac{Y(e^{j\omega})}{e^{j\omega}} = \frac{-2}{e^{j\omega}-\frac{1}{2}} + \frac{3}{e^{j\omega}-\frac{3}{4}} \quad => \quad Y(e^{j\omega}) = \frac{-2e^{j\omega}}{e^{j\omega}-\frac{1}{2}} + \frac{3e^{j\omega}}{e^{j\omega}-\frac{3}{4}}$$

Applying IDTFT

$$y(n) = -2\,(\frac{1}{2})^n u(n) + 3\,(\frac{3}{4})^n u(n)$$

Example 5.3: Find output response using Z-transform

$$y(n)-\frac{3}{2}y(n-1)+\frac{1}{2}y(n-2)=2x(n)+\frac{3}{2}x(n-1)$$

Where $y(-1)=0 \quad y(-2)=1 \qquad x(n)=\left(\frac{1}{4}\right)^n u(n)$

Solution:

$$y(n)-\frac{3}{2}y(n-1)+\frac{1}{2}y(n-2)=2x(n)+\frac{3}{2}x(n-1)$$

Taking Z-transform

$$Y(Z)-\frac{3}{2}[Z^{-1}Y(Z)+y(-1)]+\frac{1}{2}[Z^{-2}Y(Z)+Z^{-1}y(-1)+y(-2)]$$

$$=2X(Z)+\frac{3}{2}[Z^{-1}X(Z)+x(-1)]$$

$$Y(Z)-\frac{3}{2}[Z^{-1}Y(Z)]+\frac{1}{2}[Z^{-2}Y(Z)+1]=X(Z)\left[2+\frac{3}{2}Z^{-1}\right] \quad \because y(-1)=0, y(-2)=1$$

$$Since\ x(n) is\ causal\ signal\ x(-1)=0$$

$$x(n)=\left(\frac{1}{4}\right)^n u(n)\Rightarrow X(Z)=\frac{1}{1-\frac{1}{4}Z^{-1}}=\frac{Z}{Z-\frac{1}{4}}$$

$$\therefore Y(Z)\left[1-\frac{3}{2}Z^{-1}+\frac{1}{2}Z^{-2}\right]=\frac{Z}{Z-\frac{1}{4}}\left[2+\frac{3}{2}Z^{-1}\right]-\frac{1}{2}$$

$$Y(Z)=\frac{Z}{Z-\frac{1}{4}}\frac{(2+\frac{3}{2}Z^{-1})}{(1-\frac{3}{2}Z^{-1}+\frac{1}{2}Z^{-2})}-\frac{\frac{1}{2}}{1-\frac{3}{2}Z^{-1}+\frac{1}{2}Z^{-2}}$$

$$Y(Z)=\frac{Z}{Z-\frac{1}{4}}\left(\frac{2Z^2+\frac{3}{2}Z}{Z^2-\frac{3}{2}Z+\frac{1}{2}}\right)-\frac{\frac{1}{2}Z^2}{Z^2-\frac{3}{2}Z+\frac{1}{2}}$$

$$\frac{Y(Z)}{Z}=\frac{(2Z^2+\frac{3}{2}Z)}{(Z-\frac{1}{4})(Z-1)(Z-\frac{1}{2})}-\frac{\frac{1}{2}Z}{(Z-1)(Z-\frac{1}{2})}=\frac{(2Z^2+\frac{3}{2}Z)-\frac{1}{2}Z(Z-\frac{1}{4})}{(Z-\frac{1}{4})(Z-1)(Z-\frac{1}{2})}$$

$$=\frac{2Z^2+\frac{3}{2}Z-\frac{1}{2}Z^2+\frac{1}{8}Z}{(Z-\frac{1}{4})(Z-1)(Z-\frac{1}{2})}=\frac{\frac{3}{2}Z^2+\frac{13}{8}Z}{(Z-\frac{1}{4})(Z-1)(Z-\frac{1}{2})}$$

$$=\frac{A}{(Z-\frac{1}{4})}+\frac{B}{(Z-1)}+\frac{C}{(Z-\frac{1}{2})}$$

$$\frac{3}{2}Z^2+\frac{13}{8}Z=A(Z-1)\left(Z-\frac{1}{2}\right)+B\left(Z-\frac{1}{4}\right)\left(Z-\frac{1}{2}\right)+C\left(Z-\frac{1}{4}\right)(Z-1)$$

At $Z=\frac{1}{4}$: $A=\frac{8}{3}$

At $Z=1$: $B=\frac{25}{3}$

At $Z=\frac{1}{2}$: $C=\frac{-19}{2}$

$$\therefore \frac{Y(Z)}{Z}=\frac{\frac{8}{3}}{Z-\frac{1}{4}}+\frac{\frac{25}{3}}{Z-1}-\frac{\frac{19}{2}}{Z-\frac{1}{2}}$$

$$Y(Z)=\frac{8}{3}\frac{Z}{Z-\frac{1}{4}}+\frac{25}{3}\frac{Z}{Z-1}-\frac{19}{2}\frac{Z}{Z-\frac{1}{2}}$$

$$y(n) = \frac{8}{3}\left(\frac{1}{4}\right)^n u(n) + \frac{25}{3}u(n) - \frac{19}{2}\left(\frac{1}{2}\right)^n u(n)$$

Example 5.4: Convolve the following sequences using Tabulation method

$$x(n) = \frac{n}{3}\ for\ 0 \le n \le 6$$

$$h(n) = 1\ for -2 \le n \le 2$$

$$x(n) = \{\underset{\uparrow}{0}, \tfrac{1}{3}, \tfrac{2}{3}, \tfrac{3}{3}, \tfrac{4}{3}, \tfrac{5}{3}, \tfrac{6}{3}\} \qquad h(n) = \{1, \ 1, \ \underset{\uparrow}{1}, \ 1, \ 1\}$$

Tabulation method

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \quad or \quad y(n) = h(n) * x(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$

$$x(n) = \{\underset{\uparrow}{0}, \tfrac{1}{3}, \tfrac{2}{3}, \tfrac{3}{3}, \tfrac{4}{3}, \tfrac{5}{3}, \tfrac{6}{3}\} \qquad h(n) = \{1, \ 1, \ \underset{\uparrow}{1}, \ 1, \ 1\}$$

k	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9
$x(k)$					0	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{3}{3}$	$\frac{4}{3}$	$\frac{5}{3}$	$\frac{6}{3}$			
$h(k)$			1	1	1	1	1							
$h(-k)$			1	1	1	1	1							
$h(-k-2)$	1	1	1	1	1									
$h(-k-1)$		1	1	1	1	1								
$h(-k)$			1	1	1	1								
$h(-k+1)$				1	1	1	1	1						
$h(-k+2)$					1	1	1	1	1					
$h(-k+3)$						1	1	1	1	1				
$h(-k+4)$							1	1	1	1	1			
$h(-k+5)$								1	1	1	1			
$h(-k+6)$									1	1	1	1	1	
$h(-k+7)$										1	1	1	1	
$h(-k+8)$											1	1	1	1

$$y(-2) = (0)(1) = 0$$

$$y(-1) = (0)(1) + \left(\frac{1}{3}\right)(1) = \frac{1}{3}$$

$$y(0) = (0)(1) + \left(\frac{1}{3}\right)(1) + \left(\frac{2}{3}\right)(1) = 1$$

$$y(1) = (0)(1) + \left(\frac{1}{3}\right)(1) + \left(\frac{2}{3}\right)(1) + \left(\frac{3}{3}\right)(1) = 2$$

$$y(2) = (0)(1) + \left(\frac{1}{3}\right)(1) + \left(\frac{2}{3}\right)(1) + \left(\frac{3}{3}\right)(1) + \left(\frac{4}{3}\right)(1) = \frac{10}{3}$$

$$y(3) = (\frac{1}{3})(1) + (\frac{2}{3})(1) + (\frac{3}{3})(1) + (\frac{4}{3})(1) + (\frac{5}{3})(1) = 5$$

$$y(4) = (\frac{2}{3})(1) + (\frac{3}{3})(1) + (\frac{4}{3})(1) + (\frac{5}{3})(1) + (\frac{6}{3})(1) = \frac{20}{3}$$

$$y(5) = (\frac{3}{3})(1) + (\frac{4}{3})(1) + (\frac{5}{3})(1) + (\frac{6}{3})(1) = 6$$

$$y(6) = (\frac{4}{3})(1) + (\frac{5}{3})(1) + (\frac{6}{3})(1) = 5$$

$$y(7) = (\frac{5}{3})(1) + (\frac{6}{3})(1) = \frac{11}{3}$$

$$y(8) = (\frac{6}{3})(1) = 2$$

$$\therefore y(n) = \{0, \quad \frac{1}{3}, \quad \frac{1}{t}, \quad 2, \quad \frac{10}{3}, \quad 5, \quad \frac{20}{3}, \quad 6, \quad 5, \quad \frac{11}{3}, \quad 2\}$$

Example 5.5: Obtain Cascade form realization

$$y(n) - \frac{1}{4}y(n-1) - \frac{1}{8}y(n-2) = x(n) + 3x(n-1) + 2x(n-2)$$

Solution:

$$y(n) - \frac{1}{4}y(n-1) - \frac{1}{8}y(n-2) = x(n) + 3x(n-1) + 2x(n-2)$$

Taking Z-transform

$$Y(Z) - \frac{1}{4}Z^{-1}Y(Z) - \frac{1}{8}Z^{-2}Y(Z) = X(Z) + 3Z^{-1}X(Z) + 2Z^{-1}X(Z)$$

$$\frac{Y(Z)}{X(Z)} = \frac{1 + 3Z^{-1} + 2Z^{-1}}{1 - \frac{1}{4}Z^{-1} - \frac{1}{8}Z^{-2}} = \frac{(1 + Z^{-1})(1 + 2Z^{-1})}{(1 - \frac{1}{2}Z^{-1})(1 + \frac{1}{4}Z^{-1})} = H_1(Z).H_2(Z)$$

$$H_1(Z) = \frac{(1 + Z^{-1})}{(1 - \frac{1}{2}Z^{-1})} \qquad H_2(Z) = \frac{(1 + 2Z^{-1})}{(1 + \frac{1}{4}Z^{-1})}$$

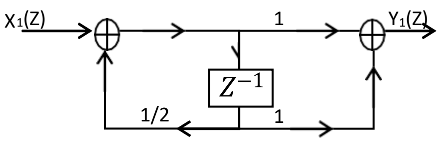


Fig 5.19 Direct form II structure of $H_1(Z)$

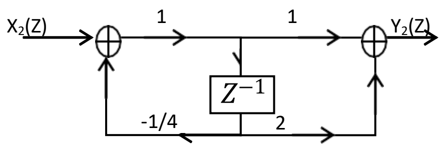


Fig 5.20 Direct form II structure of $H_2(Z)$

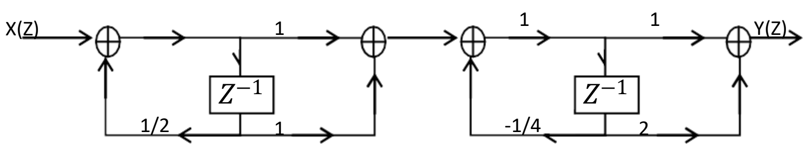


Fig 5.21 Cascade form

Example 5.6: Obtain Parallel form realization

$$y(n) - \frac{1}{4}y(n-1) - \frac{1}{8}y(n-2) = x(n) + 3x(n-1) + 2x(n-2)$$

Solution:

$$y(n) - \frac{1}{4}y(n-1) - \frac{1}{8}y(n-2) = x(n) + 3x(n-1) + 2x(n-2)$$

$$\frac{Y(Z)}{X(Z)} = \frac{1 + 3Z^{-1} + 2Z^{-2}}{1 - \frac{1}{4}Z^{-1} - \frac{1}{8}Z^{-2}}$$

$$-\frac{1}{8}Z^{-2} - \frac{1}{4}Z^{-1} + 1 \;\Big)\; 2Z^{-2} + 3Z^{-1} + 1 \;\;(\text{quotient } -16)$$

$$2Z^{-2} + 4Z^{-1} - 16$$

$$(-)\quad(-)\quad(+)$$

$$-Z^{-1} + 17$$

$$\frac{Y(Z)}{X(Z)} = -16 + \frac{-Z^{-1} + 17}{1 - \frac{1}{4}Z^{-1} - \frac{1}{8}Z^{-2}} = -16 + \frac{17 - Z^{-1}}{(1 - \frac{1}{2}Z^{-1})(1 + \frac{1}{4}Z^{-1})}$$

$$Let \quad \frac{17 - Z^{-1}}{(1 - \frac{1}{2}Z^{-1})(1 + \frac{1}{4}Z^{-1})} = \frac{A}{1 - \frac{1}{2}Z^{-1}} + \frac{B}{1 + \frac{1}{4}Z^{-1}}$$

$$17 - Z^{-1} = A\,(1 + \frac{1}{4}Z^{-1}) + B(1 - \frac{1}{2}Z^{-1})$$

$at\ Z^{-1} = -4$

$$17 + 4 = B(1 - \frac{1}{2}(-4))$$

$$\therefore B = 7$$

$at\ Z^{-1} = 2$

$$17 - 2 = A(1 + \frac{1}{4}(2))$$

$$\therefore A = 10$$

$$\therefore H(Z) = -16 + \frac{10}{1 - \frac{1}{2}Z^{-1}} + \frac{7}{1 + \frac{1}{4}Z^{-1}}$$

$$H_1(Z) = \frac{Y_1(Z)}{X_1(Z)} = \frac{10}{1 - \frac{1}{2}Z^{-1}}$$

$$\frac{Y_1(Z)}{W_1(Z)} = 10 \Rightarrow Y_1(Z) = 10W_1(Z)$$

$$\frac{W_1(Z)}{X_1(Z)} = \frac{1}{1 - \frac{1}{2}Z^{-1}}$$

$$W_1(Z) = X_1(Z) + \frac{1}{2}Z^{-1}W_1(Z)$$

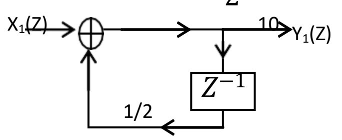


Fig 5.22 Direct form II structure of $H_1(Z)$

$$H_2(Z) = \frac{Y_2(Z)}{X_2(Z)} = \frac{7}{1 + \frac{1}{4}Z^{-1}}$$

$$\frac{Y_2(Z)}{W_2(Z)} = 7 \Rightarrow Y_2(Z) = 7W_2(Z)$$

$$\frac{W_2(Z)}{X_2(Z)} = \frac{1}{1 + \frac{1}{4}Z^{-1}}$$

$$W_2(Z) = X_2(Z) - \frac{1}{4}Z^{-1}W_2(Z)$$

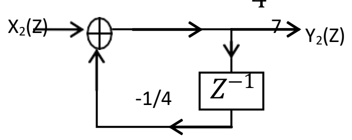


Fig 5.23 Direct form II structure of $H_2(Z)$

Combining figures Fig 5.22 and Fig 5.23 we can form parallel form realization as shown in

Fig 5.24

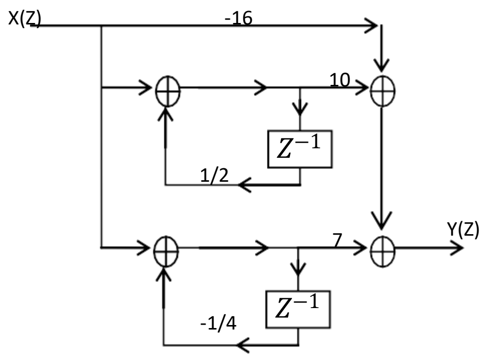
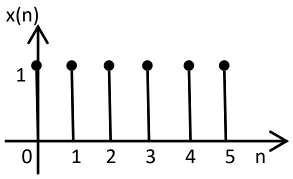


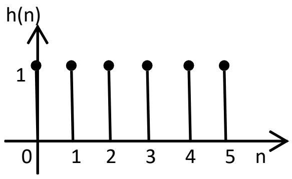
Fig 5.24 Parallel form

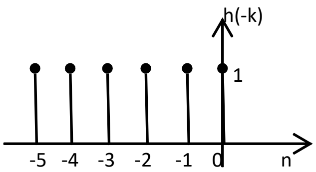
Example 5.7: Convolve the following discrete time signals using graphical convolution $x(n) = h(n) = u(n)$

Solution:

$$x(n) = u(n) = 1; n \geq 0$$
$$h(n) = u(n) = 1; n \geq 0$$

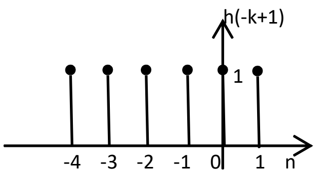




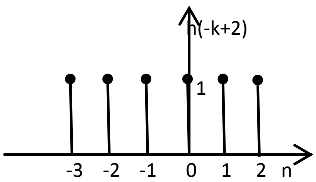


$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

$$when\, n = 0$$
$$y(0) = (1)(1) = 1$$
$$when\, n = 1$$
$$y(1) = (1)(1) + (1)(1) = 2$$



$$when\, n = 2$$
$$y(2) = (1)(1) + (1)(1) + (1)(1) = 3$$



$$\therefore y(n) = \{1,2,3,4,5, \ldots.\}$$

Example 5.8: Compute linear convolution.

$$x(n) = \{2,2,0,1,1\} \quad h(n) = \{1,2,3,4\}$$

Solution:

	2	2	0	1	1
1	2	2	0	1	1
2	4	4	0	2	2
3	6	6	0	3	3
4	8	8	0	4	4

$$y(n) = x(n) * h(n) = \{2,6,10,15,11,5,7,4\}$$